Convex relaxations for a generalized Chan-Vese model

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Abstract. We revisit the Chan-Vese model of image segmentation with a focus on the encoding with several integer-valued labeling functions. We relate several representations with varying amount of complexity and demonstrate the connection to recent relaxations for product sets and to dual maxflow-based formulations. For some special cases, it can be shown that it is possible to guarantee binary minimizers. While this is not true in general, we show how to derive a convex approximation of the combinatorial problem for more than 4 phases. We also provide a method to avoid overcounting of boundaries in the original Chan-Vese model without departing from the efficient product-set representation. Finally, we derive an algorithm to solve the associated discretized problem, and demonstrate that it allows to obtain good approximations for the segmentation problem with various number of regions.[1]

1 Introduction

In this paper we focus image segmentation formulated as a variational problem. The general problem we are interested in, is to find a partition $\{\Omega_i\}_{i=1}^n$ of the image domain Ω, by minimizing an energy functional of the form

$$\min_{\{\Omega_i\}_{i=1}^n} \sum_{i=1}^n \int_{\Omega_i} f_i(I^0(x))\,dx + \alpha R(\{\partial\Omega_i\}_{i=1}^n)$$
$$\text{s.t.} \quad \bigcup_{i=1}^n \Omega_i = \Omega, \quad \bigcap_{i=1}^n \Omega_i = \emptyset. \tag{1}$$

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Here $R(\{\partial\Omega_i\}_{i=1}^n)$ is a regularization term, and I^0 is a given input image. A popular choice for the regularizer is the Potts regularizer, which measures the total length of the region boundaries,

$$R(\{\partial\Omega_i\}_{i=1}^n) = \frac{1}{2}\sum_{i=1}^n |\partial\Omega_i|. \tag{2}$$

In order to build up algorithms for solving (1) numerically, one needs some representation of the regions in terms of functions instead of subsets. Over the last 30 years, several such representations have been proposed, including the level set method [16] and phase field method. Recently, there has been a particular interest in piecewise constant representations [12], where each regions is uniquely associated with a value of some binary or integer constrained functions. A reason for the popularity of this approach, is that very good convex relaxations often can be derived by relaxing the integrality constraints of the functions [6, 18, 11, 24, 17, 1, 3]. There are in particular three classical ways of representing multiple regions $\{\Omega_i\}_{i=1}^n$ in terms of piecewise constant functions

1. *Integer-valued* labeling function [12, 13]: $\phi : \Omega \mapsto \{1, ..., n\}$ such that $\phi(x) = i$ if $x \in \Omega_i$, $i = 1, ..., n$.

2. *Simplex-constrained* vector function [11, 24]: $v : \Omega \mapsto \Delta^n = \{v \in \mathbb{R}^n : \sum_{i=1}^n v^i = 1,\ v^i \in \{0, 1\},\ i = 1, ..., n\}$ such that

$$v^i(x) := \begin{cases} 1, x \in \Omega_i \\ 0, x \notin \Omega_i \end{cases}, \quad i = 1, \dots, n.$$

   A related variant with a different parametrization of the unit simplex was proposed in [17].
3. $m = \log_2(n)$ overlapping *binary functions*: $\phi^1, ..., \phi^m : \Omega \mapsto \{0, 1\}$ such that $x \in \Omega_i$ iff $\phi^1(x)...\phi^m(x)$ is the binary representation of integer i. This representation was pioneered in a level set framework in [21] and the resulting optimization problem is often called the Chan-Vese model. The use of binary functions for the multiphase CV model in the continuous setting was done in [13, 14, 6]. It was observed in [6] that it is possible to convex relax these binary models.

For some special problems, convex relaxations exists that have proven to be exact, meaning that global minimizers of the original non-convex problems can be obtained from minimizers of the convex relaxations. This includes in particular problems with two regions [6], the labeling function representation in case of a regularization term which is convex in ϕ [18] and the Chan-Vese model with four regions under some conditions on the data term [3].

These relaxations have been motivated by the theory of discrete optimization, where it is known that the corresponding discrete optimization problems defined over a discrete image domain are submodular and can be solved efficiently by graph based optimization algorithm such as max-flow/min-cut. However, for the majority of variational segmentation problem of interest, including (1) with Potts regularizer (2), the corresponding discrete optimization problems are non-submodular (and

actually NP-hard). Convex relaxations have been proposed for such problems that are not guaranteed to provide exact solutions in advance, but can yield good approximations in practice.
This paper aims to give an overview of different representations of the problem (1). In addition to focusing on the three approaches mentioned above, we propose new representations as combinations of several labeling functions and several simplex constrained functions.
Secondly, we derive convex relaxations for the problems based on the convex envelope of the data fidelity term. As a special case, we obtain a convex relaxation of the Chan-Vese model with an arbitrary number of regions. Up until now, global optimization algorithms for this model have only been available in case of four regions [2, 8, 3]. In contrast to other relaxations [5] with more than four regions, ours is the tightest because it is based on the convex envelope of the non-convex data term. Furthermore, the number of unknowns grow as $O(log_2(n))$ instead of $O(n)$ in [5]. A simultaneous work [15] appearing in this conference also derives a convex relaxation of the Chan-Vese model by extending the max-flow model developed in [3].
The proposed relaxations are closely related to the recent work [20], which derived a convex relaxation for vector valued labeling problems. In contrast to [20], our original problems are not vector valued. Instead, we use a vector representation to significantly reduce the number of unknowns. We also derive the relaxation without on an initial simplex constrained conversion used in [20].
We derive a set of conditions which can be checked in advance to guarantee that a global minimizer is obtained from the relaxations. While in practice these seem to hold only in very rare cases, our approach can at least produce good approximations. the best approximations that are theoretically possible. A convex relaxation for Potts regularizer is a also derived by building on the vector relaxation and the work [17, 3]. Efficient algorithms are proposed for all the problems based on Augmented Lagrangian methods.

2 Different representations of the partition as vector-valued functions

2.1 2^m regions with m binary functions

We start by focusing on the representation 3 given in the introduction, which is the binary version of the level set framework [21]. For each $i \in \{1, ..., n\}$, let $a_i^1 a_i^2 ... a_i^m$ denote the binary representation of i or any permutation of the digits in the binary representation. Define $w_0(s) := s$ and $w_1(s) := 1 - s$ and introduce m binary functions $\phi^1, ..., \phi^m : \Omega \mapsto \{0, 1\}$.
The general model in [21] could then be written in terms of polynomials in $\{\phi^i\}_{i=1}^m$ as

$$\min_{\{\phi^i\}_{i=1}^m} \int_\Omega \sum_{i=1}^n \prod_{k=1}^m w_{a_i^k}(\phi^k) f_i \, dx + \alpha \sum_{k=1}^m \int_\Omega |\nabla \phi^k| \tag{3}$$

subject to

$$\phi^i \in \mathbb{B} := \{\phi \in BV(\Omega) \,:\, \phi(x) \in \{0,1\} \text{ for a.e. } x \in \Omega\},\ i = 1, ..., m. \tag{4}$$

It is also possible to represent a number of n regions which is not a power of 2 by choosing m is the small integer such that $n < 2^m$ and setting $f_i = \infty$ for the $2^m - n$ number of excess indices i.

2.2 Product space of several labeling functions

A natural extension of the model in the previous section is to represent the image partition in terms of an *integer-valued* labeling function $\phi : \Omega \mapsto \{0, ..., n-1\}$ with the understanding that $\phi(x) = i$ if and only if $x \in \Omega_i$ for $i = 1, ..., n$. It has recently been established that such problems can be solved exactly if the regularizer is the total variation of the labeling function $R(\phi) = \int_\Omega |\nabla \phi|$ [9, 18].
This can also be combined with the approach from the previous section, by representing the partition with several labeling functions $(\phi^1, ..., \phi^m)$ taking several integer values, as proposed in a level set framework in [7]. Denote by $\mathcal{L}_i = \{0, ..., N_i - 1\}$ the set of feasible values for ϕ^i, where N_i is the number of feasible integer values and define the vector function

$$\phi = (\phi^1, ..., \phi^m) \,:\, \Omega \to \mathcal{L}_1 \times ... \times \mathcal{L}_m \subset \mathbb{Z}^m \tag{5}$$

For each $x \in \Omega$, $\phi(x)$ can take $\prod_{i=1}^m N_i$ different values and thus be used to represent $n = \prod_{i=1}^m N_i$ regions. Let $\{a^i\}_{i=1}^n$ denote an enumeration of all feasible values for ϕ, i.e., for each $i = 1, ..., n$,

$$a^i = \left(a_1^i \cdots a_m^i \right)^\top \tag{6}$$

such that $a_k^i \in \mathcal{L}_k$ for $k = 1, ..., m$. Region Ω_i can be encoded as

$$\Omega_i = \{x \in \Omega \text{ s.t. } \phi(x) = a^i\}, \quad i = 1, ..., n \tag{7}$$

However, this encoding is not unique, as the enumeration $\{a^i\}_{i=1}^n$ can be reordered in any way. There are $n!$ such reordeings and they can be formulated generally using a permutation matrix P as follows

$$[a^1 ... a^n] \leftarrow [a^1 ... a^n] \cdot P \tag{8}$$

The choice of permutation may have an effect on the quality of the relaxation. For instance [3] showed that a particular permutation of the four region model was crucial for producing exact global minimizers of the original problem. By introducing a function $f \,:\, \mathcal{L}_1 \times ... \times \mathcal{L}_m \times \Omega \mapsto \mathbb{R}$,

$$f(\phi(x), x) = \begin{cases} f_i(x), & \text{if } \phi(x) = a^i,\ i = 1, ..., n \\ +\infty, & \text{otherwise,} \end{cases} \tag{9}$$

we can define the regularized energy in terms of ϕ,

$$\min_\phi \int_\Omega f(\phi(x), x)\, dx + \alpha \sum_{i=1}^m \int_\Omega |\nabla \phi^i| \tag{10}$$

In case $N_1 = ... = N_m = 2$, the model (10) reduces to the Chan-Vese model (3). Note that due to the separable form of the regularizer, some boundaries will be counted more than once.

2.3 Product space of several simplex constrained functions

A third way to represent the regions is in terms of several simplex constrained vector functions: let $v = (v^1, ..., v^m) : \Omega \mapsto \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_m}$ be a set of unit vector functions which satisfy

$$\sum_{k=1}^{N_i} v_k^i(x) = 1, \quad v_k^i(x) \in \{0,1\}, \quad k = 1, ..., N_i \; \forall x \in \Omega \tag{11}$$

The function v with the above constraint can represent $n = \prod_{i=1}^m N_i$ regions. For every $k_1 \in \{1, ..., N_1\}, ..., k_m \in \{1, ..., N_m\}$, let $i(k_1, ..., k_m) = k_1 + \sum_{j=2}^m (\prod_{i=1}^j N_i) k_j$ be the corresponding index, then each region can be described in terms of v by

$$\Omega_{i(k_1,...,k_m)} = \{x \in \Omega \,:\, v_{k_1}^1(x) = ... = v_{k_m}^m(x) = 1\} \tag{12}$$

In order to encode the data term we define

$$f(v(x), x) = \begin{cases} f_{i(k_1,...,k_m)}(x), & \text{if } v_(^1 x) = e_{k_1}, ..., v_(^m x) = e_{k_m}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{13}$$

The general segmentation model can then be formulated as

$$\min_{v \in B} \int_\Omega f(v(x), x) + \alpha \sum_{i=1}^m \sum_{k_i=1}^{N_i} \int_\Omega |\nabla v_{k_i}^i|. \tag{14}$$

An advantage of this representation compared to (10) is that the regularization term of (14) more closely resembles the Potts regularization term (2), and in fact exactly represents it for boundaries where only one of the v^i changes. For instance, if $m = 2$, the boundaries will be counted at most twice, with the majority being counted once.

3 Convex relaxations based on the convex envelope

In this section, we derive convex approximations for the models introduced in Sect. 2 based on their convex envelopes, which are defined as the largest convex function majorized by a given function and can under very general conditions be computed by computing the Legendre-Fenchel biconjugate [19].

This process ensures that the convexified energy is close to the original function. In some cases, minimizers of the original problem are also minimizers of the convex envelope. Conditions which guarantee this property in advance are derived in Sect. 4. In [20], a convex relaxation was proposed for vector valued labeling problems of the form (10) with arbitrary data terms. Our work is an adaptation of [20] with a few distinctions. The paper [20] focused on integer-constrained vector labeling, but first converted the problem to a simplex-constrained formulation and derived the convex relaxation based on this formulation. In contrast, we derive the convex envelope directly based on the integer labeling formulation, which leads to a new convex problem with fewer unknowns and a novel integer thresholding step.

3.1 Product space of several labeling functions

The energy functional (10) is composed of a sum of a non-convex data term and a convex regularizer. We ignore the regularization term in the following derivations, since it is already convex. Since deriving the full convex envelope is in general intractable, in the following we focus on deriving the convex envelope *pointwise* at each $x \in \Omega$, for f defined as in (9):

$$\begin{aligned} f^*(p(x),x) &= \sup_{u(x)\in\mathbb{R}^m} \big\{\big(\sum_{i=1}^m p_i(x)u^i(x)\big) - f(u(x),x)\big\} \\ &= \max_{u(x)\in\{a_i\}_{i=1}^n} \big\{\big(\sum_{i=1}^m p_i(x)u^i(x)\big) - f(u(x),x)\big\} \end{aligned}$$

The biconjugate is

$$\begin{aligned} f^{**}(\phi(x),x) &= \sup_{p(x)\in\mathbb{R}^m} \{\sum_{i=1}^m \phi^i(x)p_i(x) - f^*(p(x),x)\} \\ &= \sup_{p(x)\in\mathbb{R}^m} \{\sum_{i=1}^m \phi^i p_i(x) + \min_{u(x)\in\{a_i\}_{i=1}^n} \{f(u(x),x) - \sum_{i=1}^m p_i(x)u^i(x)\}\} \\ &= \sup_{p(x)\in\mathbb{R}^m, p_0(x)\in\mathbb{R}} \sum_{i=1}^m \phi^i(x)p_i(x) + p_0(x) \qquad (15) \\ &\quad \text{s.t. } p_0(x) \le -\sum_{i=1}^n u^i(x)p_i(x) + f(u(x),x)\,, \quad \forall u(x) \in \{a_i\}_{i=1}^n. \end{aligned}$$

Note that $f^{**}(\phi(x),x) = +\infty$ if ϕ is not in the convex hull of the points, $\mathrm{conv}\{a^1,...,a^n\}$: if not, the function $f^{**}+\delta_{\mathrm{conv}\{a^1,...,a^n\}}$ is strictly greater that f^{**} but still majorized by f, which contradicts the maximality of f^{**} as the convex hull of f. The overall problem with regularization we wish to solve is therefore

$$\begin{aligned} &\min_{\phi\in BV(\Omega)} \sup_{p\in L^2(\Omega)^{m+1}} \int_\Omega p_0(x) + \sum_{i=1}^m \phi^i(x)p_i(x)\,dx + \alpha\sum_{i=1}^m \int_\Omega |\nabla\phi^i| \qquad (16) \\ &\text{s.t. } p_0(x) + \sum_{i=1}^n u^i(x)p_i(x) \le f(u(x),x)\,, \quad \forall u(x)\in\{a^i\}_{i=1}^n, \quad \forall x\in\Omega. \end{aligned}$$

We want an integral solution $\phi^1,...,\phi^m$ to the minimization problem (16). However, it cannot in general be expected that the solution is integral at every point. Therefore, we apply a thresholding procedure with parameter $t \in (0,1]$ as follows

$$(\phi^i)^t(x) = \begin{cases} \lfloor\phi^i\rfloor, & \text{if } \phi^i(x) < \lfloor\phi^i(x)\rfloor + t \\ \lceil\phi^i\rceil, & \text{else,} \end{cases} \qquad i = 1,...,m \qquad (17)$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor and ceiling functions respectively. If the constraint set is binary, i.e. $N_1 = ... = N_m = 2$, this corresponds to the standard thresholding procedure in [6].

3.2 Product space of several simplex constrained functions

In the same way, one can derive the envelope relaxation of the model (14). We start by defining the vector $p = (p^1, ..., p^m)$ such that $p^k \in \mathbb{R}^{N_k}$. Computing the envelope pointwise for each $x \in \Omega$ and ignoring the regularization term, one ends up with

$$\sup_{\{\{p_i^k(x)\}_{i=1}^{N_k}\}_{k=1}^m} \sum_{k=1}^{m} \sum_{i=1}^{N_k} p_i^k(x) v_i^k(x) \tag{18}$$

subject to

$$\sum_{i=1}^{N_k} p_{a_i^k}^k(x) \leq f_i(x), \quad k = 1, .., m, \tag{19}$$

$$\sum_{i=1}^{N_k} v_i^k(x) = 1, \quad k = k, ..., m, \tag{20}$$

$$v_i^k(x) \in [0,1], \quad i = 1, ..., N_k, \quad k = 1, ..., m. \tag{21}$$

which is also the pointwise relaxation of the data term for vector labeling problems that was proposed in [20]. The objective function (18) with constraints (19)-(21) also arises if one computes the standard LP-relaxation (review: [22]) of the combinatorial data term (14). The model (14) can therefore be relaxed as

$$\sup_{\{\{p_i^k(x)\}_{i=1}^{N_k}\}_{k=1}^m} \sum_{k=1}^{m} \sum_{i=1}^{N_k} \int_\Omega p_i^k(x) v_i^k(x)\, dx + \alpha \sum_{i=1}^{m} \sum_{k_i=1}^{N_i} \int_\Omega |\nabla v_{k_i}^i|\, dx \tag{22}$$

subject to (19)–(21) for all $x \in \Omega$. Although we cannot expect a binary solution in general, an approximate partition can be obtained through the thresholding step

$$v_k^i(x) \leftarrow \begin{cases} 1, \text{ if } i = \arg\max_j v_k^j(x) \\ 0, \text{ otherwise.} \end{cases}, \quad i = 1, \dots, n$$

Note that the Chan-Vese model can be obtained as a special case by substituting $v_1^k = 1 - \phi^k$ and $v_2^k = \phi^k$. The approach of Bae-Yuan-Tai [1] is also a special case of this general model where global minimization can be guaranteed under some moderate conditions.

4 Special cases which guarantee global minimizers

We show that under some conditions on the data term of (3), exact solutions will be produced by the relaxation (16). In particular, this is true if the data term is submodular. Observe that the energy in (3) pointwise consists of interactions between m binary variables. An energy function of two binary variables $E(\phi^1, \phi^2)$ is said to be *submodular* if [4, 10]

$$E(1,0) + E(0,1) \leq E(1,1) + E(0,0). \tag{23}$$

A higher-order function $E(x^1, x^2, x^3)$ of 3 binary variables is said to be submodular if the projections onto all functions of two binary variables are submodular [4, 10], i.e., if

$$E(b, x^2, x^3), \quad E(x^1, b, x^3), \quad E(x^1, x^2, b) \tag{24}$$

are submodular for every $b \in \{0, 1\}$.

In [3], a different relaxation was proposed for the model (3) with four regions, which was shown to be exact in case of a submodular data term. Specifically, it was shown (3) could be reformulated as

$$\min_{\phi^1, \phi^2 \in [0,1]} \alpha \int_\Omega |\nabla \phi^1| + \alpha \int_\Omega |\nabla \phi^2| \tag{25}$$

$$+ \int_\Omega (1 - \phi^1(x))C(x) + (1 - \phi^2(x))D(x) + \phi^1(x)A(x) + \phi^2(x)B(x)\, dx$$

$$+ \int_\Omega \max\{\phi^1(x) - \phi^2(x), 0\}E(x) - \min\{\phi^1(x) - \phi^2(x), 0\}F(x)\, dx$$

where $A, ..., F$ satisfy the linear system of equations

$$\begin{cases} A(x) + B(x) &= f_2(x) + \sigma(x) \\ C(x) + D(x) &= f_3(x) + \sigma(x) \\ A(x) + E(x) + D(x) &= f_1(x) + \sigma(x) \\ B(x) + F(x) + C(x) &= f_4(x) + \sigma(x) \end{cases}. \tag{26}$$

where σ is an arbitrary function. Consequently it was proved that an exact global minimizer could be obtained by thresholding any solution of the convex problem.

In the following, we show that the convex envelope relaxation (16) with $N_1 = N_2 = 2$ is equivalent to the formulation (25) in case the energy is submodular. In [3] it was observed that one possible solution of (26) is

$$\begin{aligned} &A = \max\{f_2 - f_4, 0\}, B = \max\{f_4 - f_3, 0\}, C = \max\{f_4 - f_2, 0\}, \\ &D = \max\{f_3 - f_4, 0\}, E = f_1 + f_4 - f_2 - f_3. \end{aligned} \tag{27}$$

Substituting this into the integrand of the data term of (25) yields

$$\phi^1(f_2 - f_4) + \phi^2(f_4 - f_3) + (f_1 + f_4 - f_2 - f_3)\max\{\phi^1 - \phi^2, 0\}. \tag{28}$$

for the data term. On the other hand, problem (15) is a linear program with in this case effectively 3 unknowns (note that here we use $a^0 = (1,0), a^1 = (1,1), a^2 = (0,0), a^3 = (0,1)$, so that the vector $(-1,-1,1,1)$ is in the null-space of the constraint matrix for p). Any solutions must be on corners of the constraint set, which are characterized by 3 of the constraints holding with equality, and can be computed as

$$p \in \{(f_1 - f_3, f_2 - f_1), (f_2 - f_4, f_4 - f_3), \tag{29}$$

$$(f_2 - f_4, f_2 - f_1), (f_1 - f_3, f_4 - f_3)\}. \tag{30}$$

Checking these possible solutions against the remaining constraint yields that the first two solutions are feasible and the last two solutions are infeasible iff $f_4 + f_1 - f_2 - f_3 \geqslant 0$, which is exactly the submodularity condition. Substituting these two solutions into (15) yields a compact expression for f^{**},

$$\max\{\phi^1(f_1 - f_3) + \phi^2(f_2 - f_1) + f_3, \phi^1(f_2 - f_4) + \phi^2(f_4 - f_3) + f_3\}, \quad (31)$$

which can be shown to be equivalent to (28), and consequently established the equivalence of (16) and (25) for submodular data.
It can also be shown that if the 8 region model is submodular, the relaxation satisfies the coarea formula and can therefore be thresholded while preserving the energy. More details will be provided in an extended version of this paper.

5 Potts regularization term

The model (1) does not correspond exactly to the Potts regularizer (2) because some of the region boundaries are counted multiple times. In the following, we present a way to derive a convex relaxation of the Potts regularization term by using the above representation as binary functions by introducing additional constraints on the dual variables.
This relaxation is inspired by the work of [17], which derived a convex relaxation of Potts model based on the labeling function representation of the partition. An important distinction is that the number of dual constraints in our relaxation grow as $O(\log_2(n)^2)$, whereas the number of constraints in [17] grow as $O(n^2)$. The relaxation was proposed for the case of 4 regions in [3], our contribution here is the generalization to 2^m regions.
The model (16) can be written with dual variables as

$$\min_{\phi \in BV(\Omega)} \sup_{p \in L^2(\Omega)^{m+1}, q^i \in C_\alpha} \int_\Omega p_0(x) + \sum_{i=1}^{m} \phi^i(x) p_i(x) + \phi^i(x) \operatorname{div} q^i(x)\, dx \quad (32)$$

$$\text{s.t. } p_0(x) + \sum_{i=1}^{n} u^i(x) p_i(x) \le f(u(x), x)\,, \quad \forall u(x) \in \{a_i\}_{i=1}^n, \quad \forall x \in \Omega$$

where the constraint set C_α is defined as

$$C_\alpha = \{q \in C^\infty(\Omega)^{\dim(\Omega)} \ : \ |q|_\infty \le \alpha\}. \quad (33)$$

where $|q|_\infty = \sup_{x \in \Omega} |q(x)|_2$. We are now interested in the binary case, i.e. $N_1 = ... = N_m = 2$. The convex relaxation for Potts model consists of the optimization problem (32) with the extra dual constraint set

$$(q^1, ..., q^m) \in C^P \quad (34)$$

$$= \left\{ \{q^i\}_{i=1}^m \in C_\alpha \ : \ |q^i - q^j|_\infty \le \alpha, \ |q^i + q^j|_\infty \le \alpha\ ; \ \forall\, i < j \in \{1, ..., m\} \right\}.$$

If the functions $\phi^1, ..., \phi^m$ are binary, one can easily check that the last term of (32) corresponds to the Potts regularizer (2). The constraint set (34) contains $(\log_2(n))^2$ inequalities.

6 Algorithms

We derive algorithms for the problems (16) and (22). In this section, $(a^i)^k$ denotes component i of vector a at iteration k, and a^k denotes vector a at iteration k.
Define the set

$$C_p^1(x) = \{p(x) \in \mathbb{R}^m, p_0(x) \in \mathbb{R} \, : \\ p_0(x) + \sum_{j=1}^{m} u_j(x)p_j(x) \leq f(u(x),x)\,, \ \ \forall u(x) \in \{a_i\}_{i=1}^n\}$$

Applying the dual formulation of total variation (32) and rearranging the terms, the model (16) can be reformulated as

$$\min_{\phi \in BV(\Omega)} \sup_{p \in L^2(\Omega)^{m+1}, q \in C_\alpha} \int_\Omega p_0(x) + \sum_{i=1}^{m} \phi^i(x)(p_i(x) + \operatorname{div} q^i(x))\, dx$$

$$\text{s.t. } p_0(x) + \sum_{i=1}^{n} u^i(x)p_i(x) \leq f(u(x),x)\,, \ \ \forall u(x) \in \{a^i\}_{i=1}^n, \quad \forall x \in \Omega$$

Observe that ϕ^i can be interpreted as an unconstrained Lagrange multiplier for the constraint $p_i + \operatorname{div} q^i = 0$. Consequently, it possible to form the augmented Lagrangian functional

$$L(p,q,\phi) = \int_\Omega p_0 + \sum_{i=1}^{m} \phi^i(p_i + \operatorname{div} q^i)\, dx - \frac{c}{2}\sum_{i=1}^{m} ||p_i + \operatorname{div} q^i||^2 \quad (35)$$

and solve the problem (16) by the augmented Lagrangian method as follows: initialize the starting points p^0, q^0, ϕ^0, and iterate, for $k = 0, 1, ...,$

$$(p)^{k+1} = \arg\max_{(p) \in C_p^1} L(p, q^k, \phi^k),$$

$$(q^i)^{k+1} = \arg\max_{q^i} L(p^{k+1}, q^i, \phi^k), \qquad i = 1, ..., m$$

$$(\phi^i)^{k+1} = (\phi^i)^k - c((p^i)^{k+1} + \operatorname{div}(q^i)^{k+1}), \qquad i = 1, ..., m$$

The first subproblem can be solved approximately as

$$(p)^{k+1} = \Pi_{C_p^1}(p^k + \delta_p \frac{\partial L}{\partial p}(p^k, q^k, \phi^k))$$

where $\frac{\partial L}{\partial p_0}(p^k, q^k, \phi^k) = 1$ and $\frac{\partial L}{\partial p_i}(p^k, q^k, \phi^k) = {\phi^i}^k - c(p_i^k - \operatorname{div} {q^i}^k)$, $i = 1, ..., m$. The projection $\Pi_{C_p^1}$ onto C_p cannot be computed in closed form, but can be computed iteratively by Dykstra's algorithm. The subproblem involving q can be solved iteratively by computing an ascent step followed by a simple projection onto C_α. In case of a Potts regularizer, we use Dykstra's algorithm to project onto q onto its constraint set as in [17, 3]. Note that (35) resembles a max-flow problem where $p_i + \operatorname{div} q^i = 0$ is the flow conservation constraint. The algorithm above is in the same spirit as recently proposed continuous max-flow algoirthms [23, 3] which have demonstrated to be very efficient in practice.
A similar Augmented Lagrangian algorithm can also be derived for (22). To save space, we skip the details here.

7 Experiments

In the experiments we have chosen the data term

$$f_i(x) = |I(x) - c_i|^\beta \quad \forall x \in \Omega, \quad i = 1, ..., n \tag{36}$$

with $\beta = 2$. The optimal parameters c_i, $i = 1, ..., n$ are calculated by iteratively minimizing for the regions and c_i until convergence. In Figure 2(f) they are chosen uniformly between 0 and 1 without subsequent updating. We use the relaxation (16) to find partition into various number of regions. Results with 8 regions, represented by 3 binary functions ($N_1 = N_2 = N_3 = 2$), are depicted in Figure 2 and 4. Results with 6 regions, which was represented by one binary function and one function taking 3 integer values ($N_1 = 2, N_2 = 3$), are shown in Figure 3 (a) and (b). Results with 16 regions in Figure 3 (c), represented by 4 binary functions ($N_1 = ... = N_4 = 2$) . Observe that the solutions $\phi^1, ..., \phi^m$ are binary/integer at most points. In order to produce a fully binary solution we threshold according to (17). To visualize the results, we have plotted $\phi^1, ..., \phi^3$ before thresholding . The results can also be depicted in a single image by the construction $I = c_i$ in Ω_i, $i = 1, ..., n$. We also depict I before thresholding, by using the polynomial in (3) to represent the regions in a soft manner before thresholding.

In the experiments with 8 regions, it is interesting to ask whether the submodularity conditions (24) can be satisfied at every point for a permutation of the labels, which would guarantee that a global minimizer can be obtained after thresholding. Unfortunately, this was not the case for any of the possible permutations (8) of the representation, in contrast to the 4-region case, where this condition often holds [3].

This means that in most cases it is not possible to guarantee that a global minimizer will be obtained a priori. However the results seem to be good in practice and are in any case the best approximations that can be obtained using a local relaxation, i.e., a relaxation of the integrand.


(a)


(b)

Fig. 1. Test images

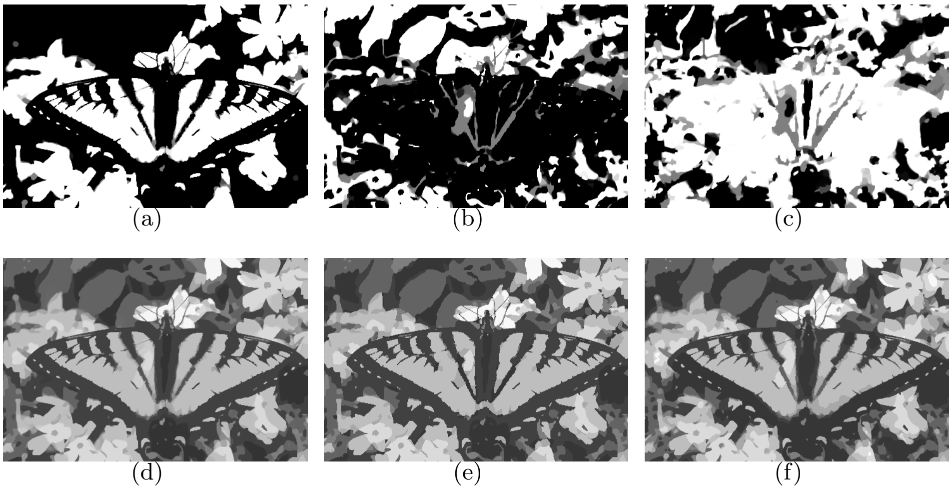

Fig. 2. 8 regions ($N_1 = N_2 = N_3 = 2$), $\alpha = 0, 1$: (a)-(c) $\phi^1, ..., \phi^3$ before threshold, (d) I before threshold, (e) I after threshold, (f) Potts relaxation after threshold

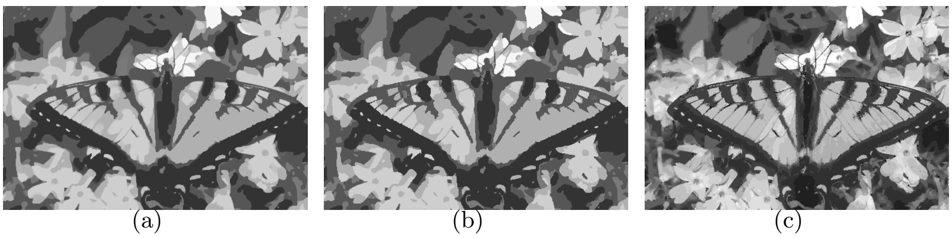

Fig. 3. (a) - (b) 6 regions ($N_1 = 2, N_2 = 3$), $\alpha = 0, 1$, (a) before threshold, (b) after threshold. (c) 16 regions ($N_1 = ... = N_4 = 2$)

Fig. 4. 8 regions, $\alpha = 0.1$. (a) Thresholded solution, (b)-(d) $\phi^1, ..., \phi^3$ before threshold

8 Conclusions

We have summarized and generalized different representations of the regions in variational image segmentation models in terms of vector functions. Convex relaxations have been developed based on the convex envelope and connected to recent relaxations for product sets and to dual maxflow-based formulations. The relaxations contain a significantly lower number of unknowns than there are regions and are the tightest convex approximations that exist for the given set of problems. Efficient algorithms have been developed and experiments have demonstrated that good approximations for the segmentation problems can be obtained in practice.

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