Asymmetric regularization of higher-order derivatives

Rob Tovey, Jan Lellmann

Institute of Mathematics and Image Computing (MIC)
University of Lübeck

MIC Institute of Mathematics and Image Computing / UNIVERSITY OF CAMBRIDGE

GAMM 2016, Braunschweig

Problem formulation

Given data I, find the (image-)information u so that

$$I = T(u) + n,$$

T "forward model", I measurements, n random noise

$\xrightarrow{T(u)+n}$

Problem

Variational Approach

$$\min_{u \in X(\Omega,\mathbb{R})} D(T(u) - I) + R(u), \qquad D \text{ data term}, \qquad R \text{ regularizer}$$

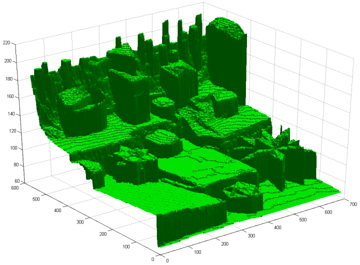
Regularizers

- $R(u) = \int_\Omega \|\nabla u\|^2 dx$ – H^1 [Whittaker'23] [Tikhonov'63]
- $R(u) = |Du|(\Omega) \approx \int_\Omega \|\nabla u\| dx$ – TV
  [RudinOsherFatemi'92], Huber [Charbonnier et al.'94] [Vese'01] various norms [SapiroRingach'96] [BlomgrenChan'98] [GoldlueckeStrekalovskiyCremers'12], q-norms [Moellenhoff et al.'15]
- $R(u) = \int_\Omega \|\nabla^2 u\|^2 dx$ – H^2 [DidasWeickertBurgeth'09]
- $R(u) = \|D^2 u\|(\Omega) \approx \int_\Omega \|\nabla^2 u\| dx$ – BH, TV^2 [Demengel'84] [LysakerLundervoldTai'03] [HinterbergerScherzer'05] [Scherzer'09], mixtures [ChambolleLions'97] [PapafitsorosSchönlieb'14]
- $R(u) = \inf_w \alpha\|Du - w\| + \beta\|Ew\| dx$ – TGV, inf-c [BrediesKunischPock'10] [SetzerSteidlTeuber'11]
- nonlocal [BourgainBrezisMironescu'01] [GilboaOsher'08] [L. et al.'14]

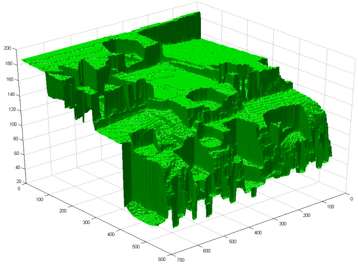
Almost always: $R(u) = R(-u)$.

[HinterbergerScherzer'05] [EkelandMoreno-Bromberg'08]

Reality

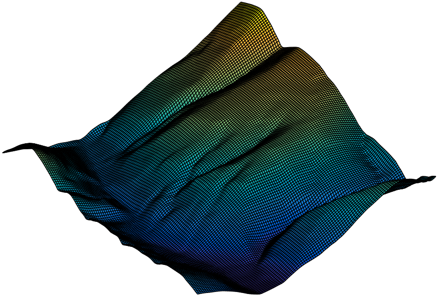


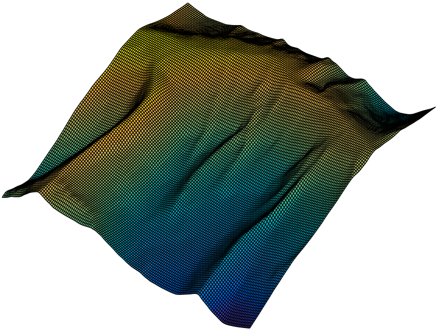
220
200
180
160
140
120
100
80
60
600
500
400
300
200
100
0
0
100
200
300
400
500
600
700

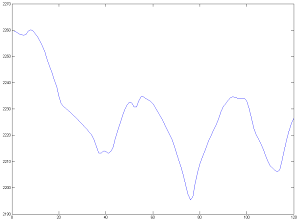


200
180
160
140
120
100
80
60
40
20
0
100
200
300
400
500
600
700
0
100
200
300
400
500
600

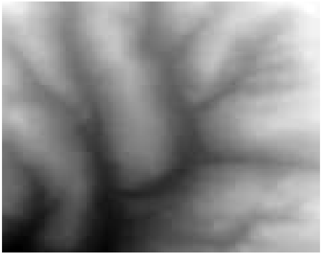
Reality

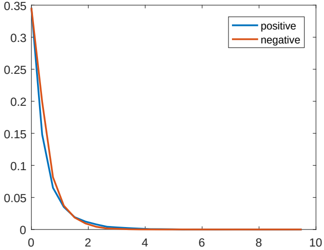




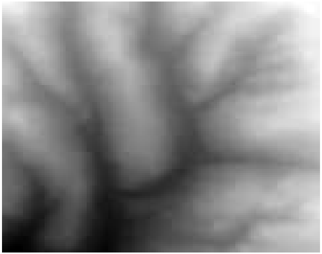

2270
2260
2250
2240
2230
2220
2210
2200
2190
0
20
40
60
80
100
120

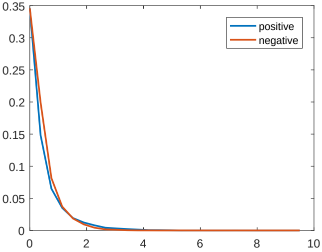
Statistics

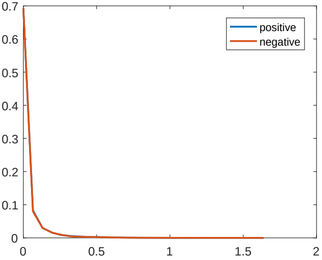




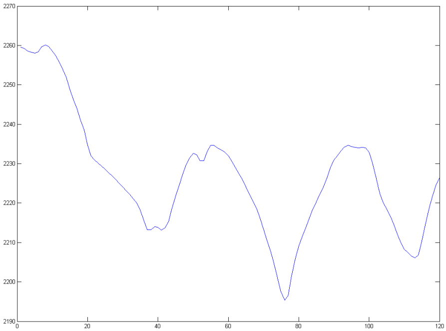
Statistics




0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
positive
negative


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
positive
negative

Which order?

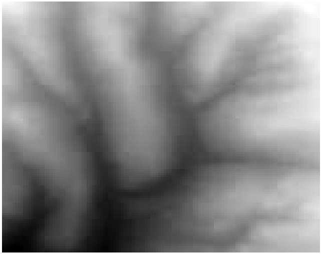


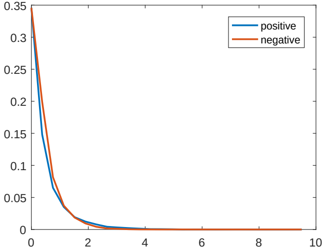
- first order:

$$\nabla(-u) = -\nabla u = \nabla(u \circ R),$$

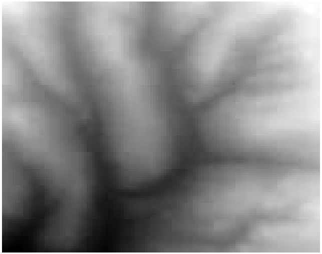
  $\rightarrow$ conflicts rotation invariance
- $\rightarrow$ second order!

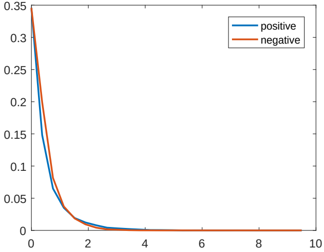
Statistics



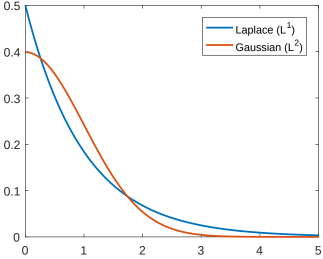


Statistics





positive
negative
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10

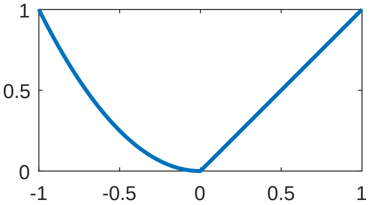


Laplace (L^1)
Gaussian (L^2)
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5

Let's make it interesting

Asymmetric higher-order $L^1 - L^2$ regularization

$$J(u) = \int_\Omega dh(D^2 u), \quad h(H) := \sum_{i=1}^{d} g(\sigma_i), \quad g(t) := \begin{cases} \alpha|t|, & t \geqslant 0, \\ \beta\frac{1}{2}t^2, & t \leqslant 0. \end{cases}$$



Let's make it interesting

Asymmetric higher-order $L^1 - L^2$ regularization

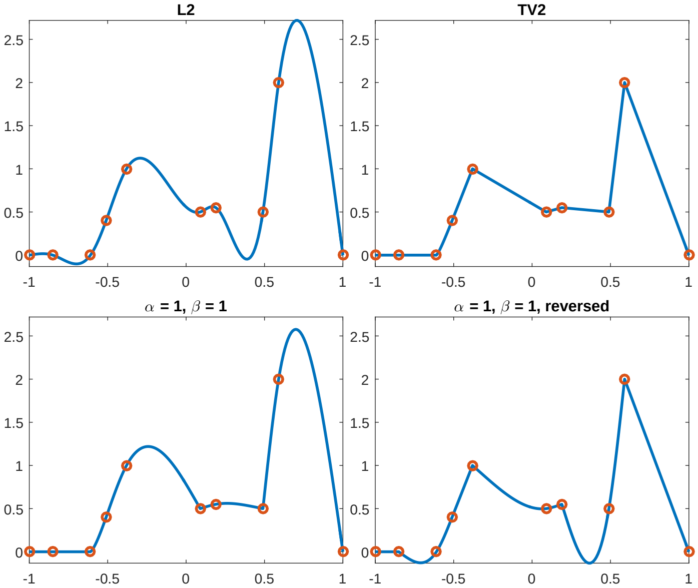
$$J(u) = \int_\Omega dh(D^2 u), \quad h(H) := \sum_{i=1}^{d} g(\sigma_i), \quad g(t) := \begin{cases} \alpha|t|, & t \geqslant 0, \\ \beta \frac{1}{2} t^2, & t \leqslant 0. \end{cases}$$

- Define as relaxation of $\int_\Omega h(\nabla^2 u) dx$ on BV^2 ($L^1 \to$ Radon measures)
- Can show:

$$J(u) = \begin{cases} +\infty, & \text{if } D_s^2 u / |D_s^2 u|(x) \not\geq 0 \text{ on a set that} \\ & \text{is not } \mathcal{H}^{d-1}\text{-negligible}, \\ \frac{\beta}{2} \int_\Omega h(\nabla^2 u) dx + \alpha |D_s^2 u|_\sigma (S_{\nabla u}), & \text{otherwise.} \end{cases}$$

- The set $\{u \in BV^2(\Omega) | J(u) < +\infty\}$ is *not* a vector space!

Interpolation



Existence

Definition (Asymmetric $L^1 - L^2$-based denoising)

For given $I \in L^2(\Omega)$, define $f : L^1(\Omega) \to \bar{\mathbb{R}}$ as

$$f_{ROF^2}(u) := \frac{1}{2} \int_\Omega (u(x) - I(x))^2 dx + J(u),$$

with $f(u)$ taking the value $+\infty$ if any of the integrals are not finite.

Proposition

Assume $d \geqslant 2$ and $\Omega \in \mathbb{R}^d$ be an open, bounded, connected extension domain. Then the minimization problem

$$\inf_{u \in \mathrm{BV}^2(\Omega)} f_{ROF^2}(u).$$

admits a minimizer $u \in \mathrm{BV}^2(\Omega) \cap L^2(\Omega)$.

Implementation

Discretized problem

$$\min_{u \in \mathbb{R}^n} D(u) + \sum_{i=1}^{n'} h(H_i u).$$

- How to solve?
  - non-smooth
  - eigenvalue-based
  - combination of quadratic and linear penalty (no pure QP/SOCP)

SDP formulation [VandenbergheBoyd'96] [Ben-TalNemirovski'01] [GrantBoyd'08]

Proposition (semidefinite reformulation)

For symmetric $H \in \mathrm{Sym}^{d \times d}$,

$$h(H) = \min_{Z_1, W_1, Z_2, W_2 \in Sym^{n\times n}} \alpha \operatorname{tr} Z_1 + \frac{\beta}{2} \operatorname{tr} Z_2 \tag{1}$$

subject to:

$$Z_1 - (H + W_1) \succeq 0$$

$$Z_1 + (H + W_1) \succeq 0$$

$$W_1 \succeq 0$$

$$\begin{pmatrix} Z_2 & H - W_2 \\ (H - W_2)^\top & I \end{pmatrix} \succeq 0$$

$$W_2 \succeq 0$$

First-order methods [ZhuChan'08] [EsserZhangChan'09] [ChambollePock'11] [Lewis'96]

Proposition (reduction of proximal steps)

Define

$$h^{+}(H) := \begin{cases} h(H), & H \in \mathrm{Sym}^{d\times d}, \\ +\infty, & \textit{otherwise.} \end{cases}$$

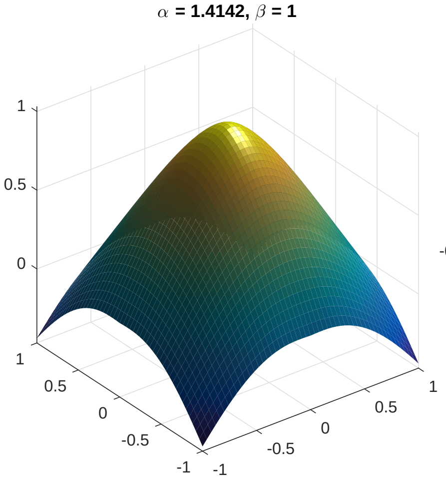
For given $H' \in \mathbb{R}^{d\times d}$, $\tau > 0$, *let* $H'' := \frac{1}{2}(H' + H'^{\top}) \in \mathrm{Sym}^{d\times d}$ *have the eigenvalue decomposition*

$$H'' = U^{\top} \operatorname{diag}(\sigma''_1, \ldots, \sigma''_d) U$$

with unitary U*. Then*

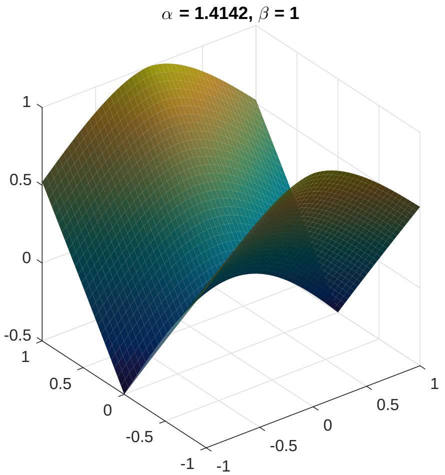
$$\operatorname{prox}_{\tau h^{+}}(H') \quad = \quad U^{\top} \operatorname{diag}(\operatorname{prox}_{\tau g}(\sigma''_1), \ldots, \operatorname{prox}_{\tau g}(\sigma''_d)) U.$$

Results

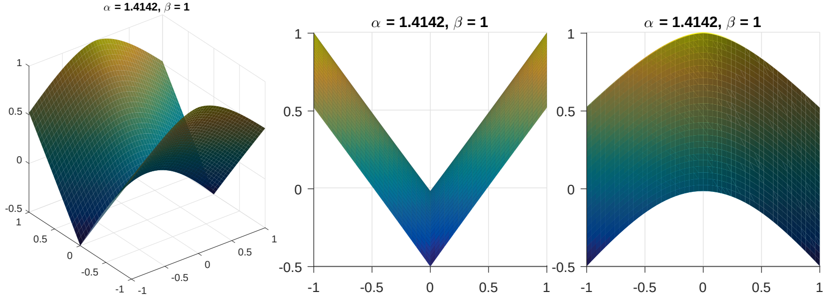


α = 1.4142, β = 1
1
0.5
0
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
α = 1.4142, β = 1
0
-0.5
-1
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

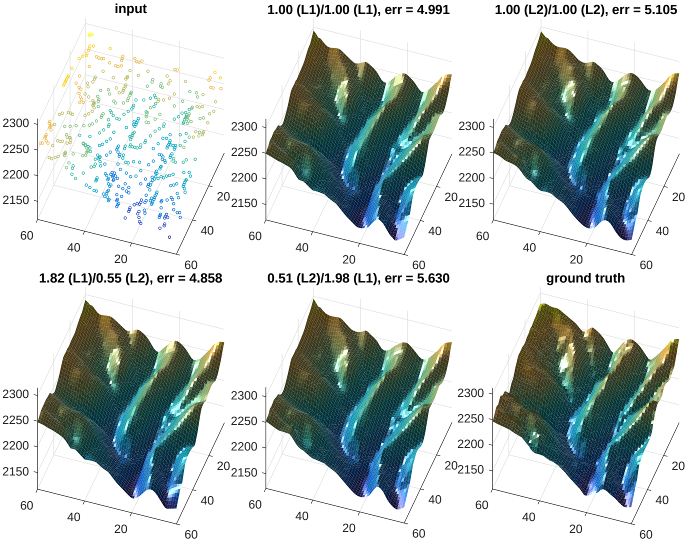
Results



Results



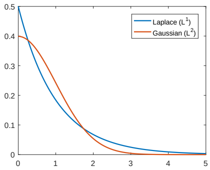
Results



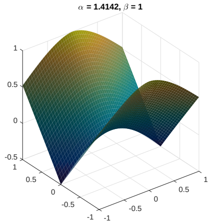
input
1.00 (L1)/1.00 (L1), err = 4.991
1.00 (L2)/1.00 (L2), err = 5.105
1.82 (L1)/0.55 (L2), err = 4.858
0.51 (L2)/1.98 (L1), err = 5.630
ground truth
2300
2250
2200
2150
60
40
20

Summary

- **Asymmetric $L^1 - L^2$ regularization**
  - eigenvalue-based
  - statistical motivation
  - second order for rotation invariance



- **Properties**
  - existence
  - SDP and first-order representation
  - it does make a difference!



Asymmetric regularization of higher-order derivatives

Rob Tovey, Jan Lellmann

Institute of Mathematics and Image Computing (MIC)
University of Lübeck

MIC Institute of Mathematics and Image Computing / UNIVERSITY OF CAMBRIDGE

GAMM 2016, Braunschweig

Extensions

Total Generalized Varation [BrediesKunischPock'10] [BrediesKunischValkonen'13]

Cascading formulation:

$$TGV^2(u) = \inf_w \alpha \int_\Omega d\|Du - w\| + \beta \int_\Omega d\|Ew\|$$

- E symmetrized distributional derivative
- allows discontinuities

New: Asymmetric Total Generalized Varation

$$TGV_J^2(u) = \inf_w \alpha \int_\Omega d\|Du - w\| + \int_\Omega dh(Ew)$$