Variational problems with finite and infinite label spaces

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Overview

Variational problems

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The problem

Given data f, find the image information u that solves

$$f = T(u) + n,$$

where T models the relation between u and f, and n is noise: Minimise

$$\min_{u:\Omega\to Y} \left\{ \underbrace{D(T(u); f)}_{\substack{\text{data-/fidelity term, compatibility} \\ \text{with measurements } f}} + \underbrace{J(u)}_{\substack{\text{regulariser, prior knowledge} \\ \text{(specific to problem)}}} \right\}$$

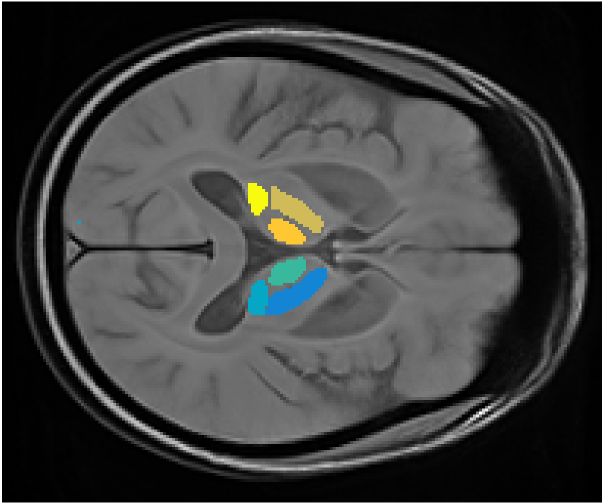
$\xrightarrow{T}$

Overview

Finite label spaces

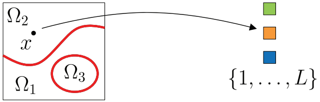
Labelling problems

- In many interesting problems the range is discrete: A discrete decision is required in every $x \in \Omega$.



with: V. Corona, C. Schönlieb, J. Acosta-Cabronero, P. Nestor/DZNE Magdeburg

Labelling problems



- Variational formulation:

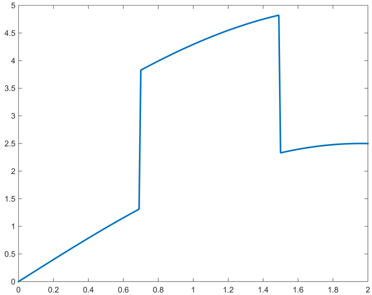
$$\min_{u:\Omega\to\{1,\ldots,L\}} \underbrace{\int_\Omega s(u(x),x)dx}_{\text{local data term}} + \underbrace{J(u)}_{\text{regulariser}}$$

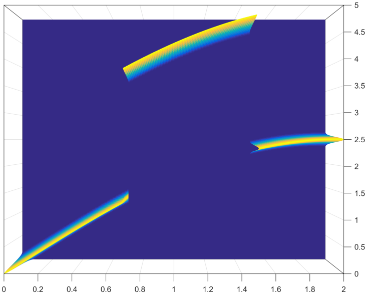
- Combinatorial/geometrical constraints $\to$ **nonconvex!**

Relaxation

- Replace the function $u : \Omega \to Y$ by the map $u' : \Omega \to \mathcal{M}(Y)$ from Ω into the set of **Dirac/point measures** on Y (vector space!):

$$u(x) = y \quad \longleftrightarrow \quad u'(x) = \delta_y .$$



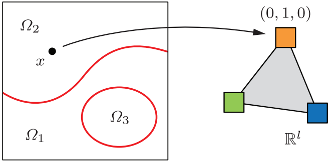


- **Relax** to the set of all maps $u' : \Omega \to \mathbb{P}(Y)$ into the set of **probability measures** on Y – always convex!

[Young 1937 – Young measures] [Schwartz 1951, de Rham 1955, Federer 1969 – currents] [Brakke 1991, Alberti, Bouchitté, DalMaso 2001 – paired calibrations]

Relaxation – finite labels

- **Finite case** $Y = \{1, \ldots, L\}$: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]



- Probability measures parametrised by unit simplex in $\mathbb{R}^L$:

$$\mathbb{P}(Y) := \{x \in \mathbb{R}^L | x \geqslant 0, \sum_i x_i = 1\},$$

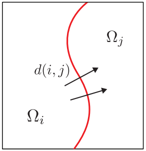
$$\min_{u' \in \mathrm{BV}(\Omega, \mathbb{P}(Y))} f(u'), \quad f(u') := \int_\Omega \langle u'(x), s(x) \rangle dx + J(u')$$

- **Convex** problem. Question: How to formulate/extend regulariser?

J. Lellmann, F. Becker, C. Schnörr, ICCV 2009

Relaxation – regulariser

- Length-based regularisation: Boundary length weighted by **interaction potential** (metric) $d(i,j)$ depending on the labels:



- $J(u)$ implicitly defined by (local) **convex envelope:**

[ChambolleCremersPock08,LellmannSchnoerr10]

$$J(u) := \sup_{v \in \mathcal{D}} \int_\Omega \langle u', \operatorname{Div} v \rangle dx = \int_\Omega \underbrace{\sigma_{\mathcal{D}_{\text{loc}}}(Du')}_{\Psi(Du')},$$

$$\mathcal{D} := \{ v \in (C_c^\infty)^{d \times L} | v(x) \in \mathcal{D}_{\text{loc}} \, \forall x \in \Omega \},$$

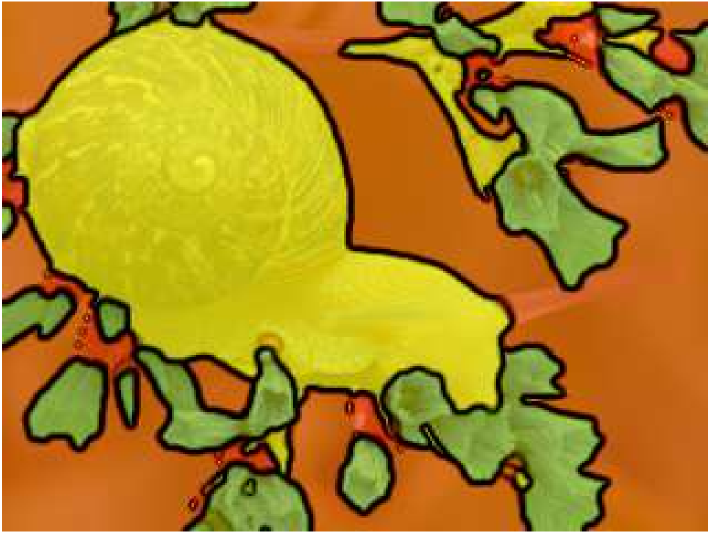
$$\mathcal{D}_{\text{loc}} := \{ (v^1, \ldots, v^L) \in \mathbb{R}^{d \times L} | \| v^i - v^j \| \leqslant d(i,j) \, \forall i,j \}.$$

- Overall **convex, non-smooth** problem

J. Lellmann, C. Schnörr, SIAM J. Imaging Sci. 2010

Example

Example



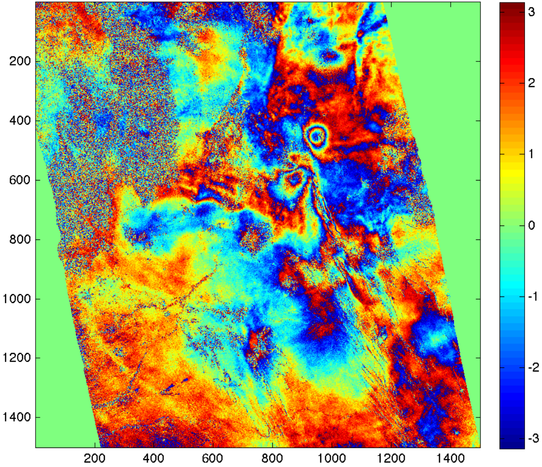
Example

Overview

Infinite label spaces

Manifold-valued problems

- Interferometric data (InSAR) contains only the *phase* ($d \mod 2\pi$) of the distance:



Manifolds – Rudin-Osher-Fatemi

- The **range** of the data u is constrained to a **(Riemannian) manifold** $\mathcal{M}$. Generalised Rudin-Osher-Fatemi:

$$\min_{u:\Omega\to\mathcal{M}} \int_\Omega d_{\mathcal{M}}(u(x), f(x))^2 dx + \underbrace{J(u)}_{\approx\int_\Omega \|Du\| dx},$$

Total Variation (TV)-based: [Giaquinta,Mucci]

$$J(u) = TV_{\mathcal{M}}(u) = \int_{\Omega\setminus S_u} |\nabla u|\, dx + \int_{S_u} d_{\mathcal{M}}(u^-, u^+)\, d\mathcal{H}^{m-1} + J_C(u).$$

- Nonconvex due to the constraints!
- Nonconvex optimisation on manifolds:

[Absil, Mahony, Sepulchre'07 – smooth] [WeinmannDemartStorath14, Bergman et al.'14 – nonsmooth]

Manifolds – approach

- Again $u'(x)$ is a *probability measure* on $\mathcal{M}$:

$$\min_{u':\Omega\to\mathbb{P}(\mathcal{M})} \sup_{p:\Omega\times\mathcal{M}\to\mathbb{R}^m} \int_\Omega \langle u', s\rangle dx + \lambda \int_\Omega \langle u', \operatorname{Div} p\rangle dx$$

$$\text{s.t. } \|p(x,z_1) - p(x,z_2)\|_2 \leqslant d_{\mathcal{M}}(z_1,z_2),\ \forall z_1, z_2 \in \mathcal{M}, \forall x \in \Omega$$

- Replace *Lipschitz constraint* by gradient-based formulation:

$$\ldots \text{s.t. } \|D_z p(x,\cdot)\|_\sigma \leqslant 1, \quad \forall z \in \mathcal{M}, \forall x \in \Omega,$$

$\|\cdot\|_\sigma$ spectral norm.

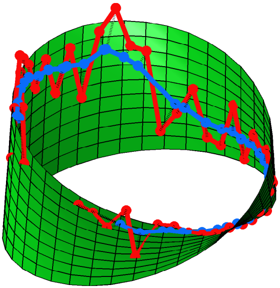
- Convex, nonsmooth, uses **manifold structure.**

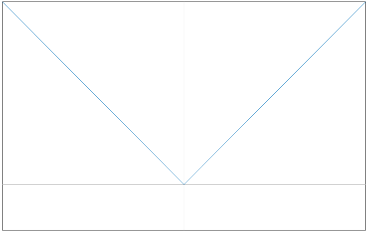
J. Lellmann, E. Strekalovskiy, S. Koetter, D. Cremers, ICCV 2013

Generalised Rudin-Osher-Fatemi

- $L^2 - TV$ (Rudin-Osher-Fatemi)

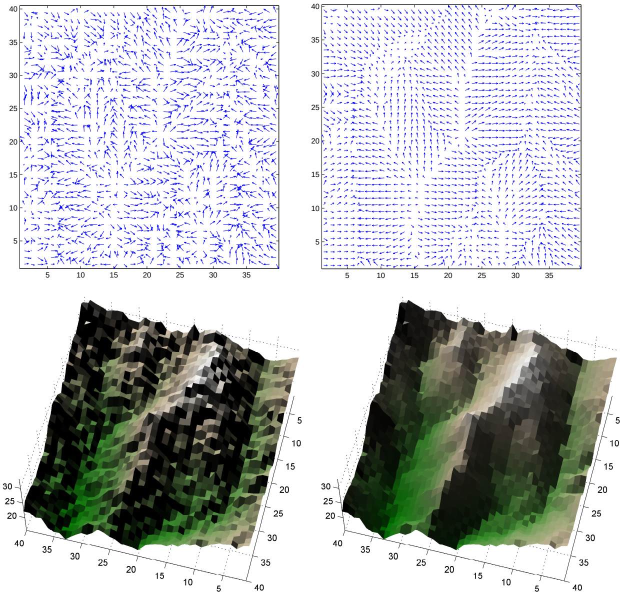
$$\min_{u:\Omega\to\mathcal{M}} D(u) + \lambda TV_{\mathcal{M}}(u)$$



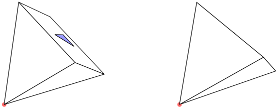


- properties similar to scalar case – contrast loss, jump preservation,. . .

Application – surface normals



Application – orientations in SO(3)



J. Lellmann, E. Strekalovskiy, S. Koetter, D. Cremers, ICCV 2013

General convex regularisers

- General convex regularization:

$$\min_{u:\Omega\to\mathcal{M}} D(u)+\int_{\Omega\setminus S_u} h(x,\nabla u)dx+\lambda\int_{S_u}\min\{\gamma, d_{\mathcal{M}}(u^-,u^+)\}d\mathcal{H}^{m-1}.$$

- Approximation: [cf. StrekalovskiyChambolleCremers2012]

$$\min_{u':\Omega\to\mathbb{P}(\mathcal{M})} \max_{\substack{p:\Omega\times\mathcal{M}\to\mathbb{R}^m\\ q:\Omega\times\mathcal{M}\to\mathbb{R}}} \int_\Omega \langle u', s\rangle dx+\lambda\int_\Omega\langle u', \operatorname{Div} p - q\rangle dx$$

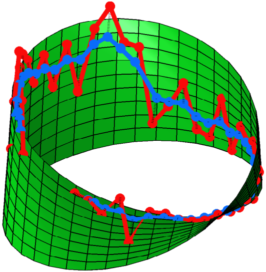
$$\text{s.t. } \|p(x,z_1)-p(x,z_2)\|_2 \leqslant d_{\mathcal{M}}(z_1,z_2), \quad \forall z_1,z_2\in\mathcal{M}, \forall x\in\Omega,$$
$$q(x,z)\geqslant h^*(x, D_z p(x,z)), \quad \forall z\in\mathcal{M},\forall x\in\Omega,$$

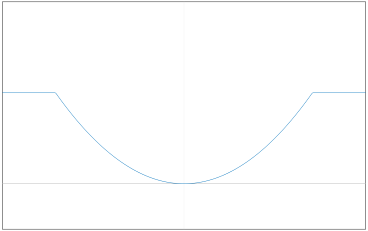
with: E. Strekalovskiy, D. Cremers, 2015

General convex regularisers

- Approximation of Mumford-Shah model on manifolds:

$$\min_{u:\Omega\to\mathcal{M}} D(u) + \lambda \int_{\Omega\setminus S_u} \|\nabla u(x)\|^2 dx + \gamma \mathcal{H}^{m-1}(S_u)$$





Bregman

- **Bregman iteration** [Osher, Burger, Goldfarb, Xu, Yin'05] – "convexity splitting"

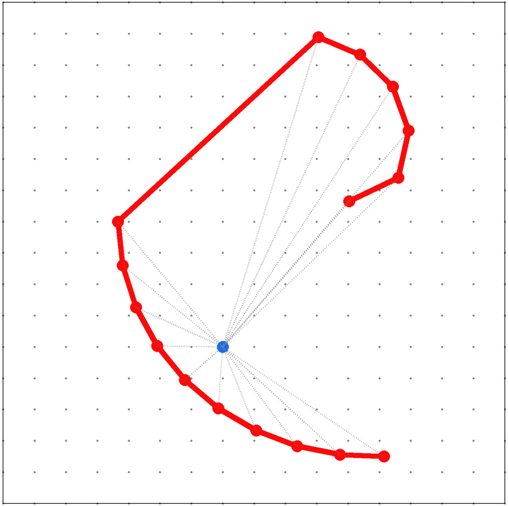
$$\min_u D(u) + 0 = \min_u D(u) + \underbrace{J(u)}_{convex} \underbrace{-J(u)}_{concave}$$

- *Linearise* the concave part:

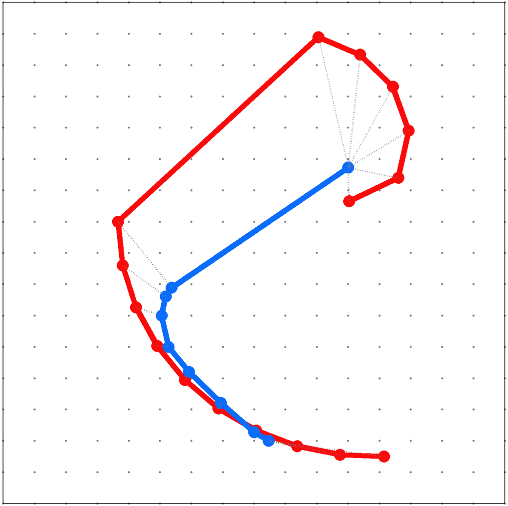
$$u^{k+1} = \arg\min_u D(u) + J(u) - (J(u^k) + \langle v^k, u - u^k \rangle), \quad v^k \in \partial J(u^k)$$

- On scalar data, gradually introduces details and converges to the input → stop if suitable solution found
  - scalar case: $u(x)$ are **real values**
  - here: $u(x)$ are **probability distributions**

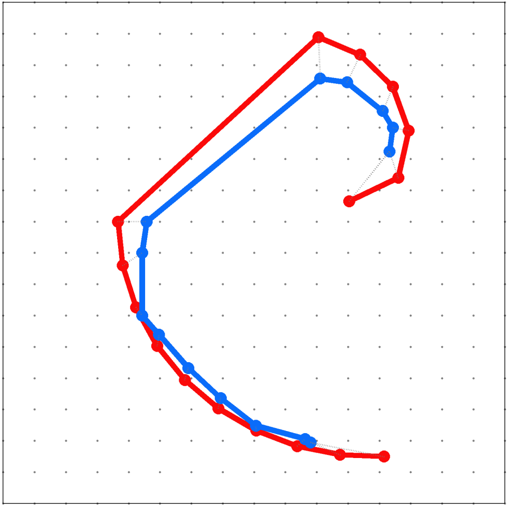
Bregman



Bregman



Bregman



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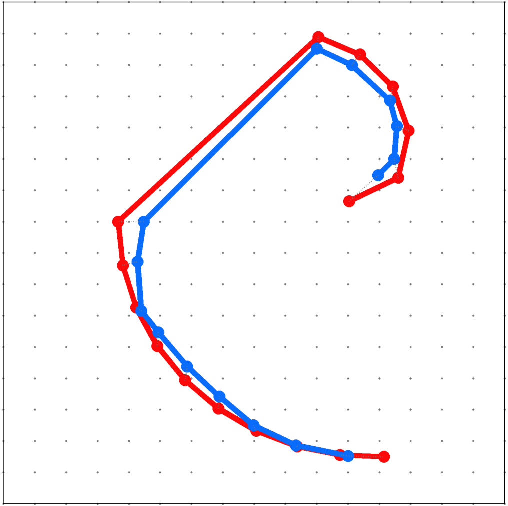
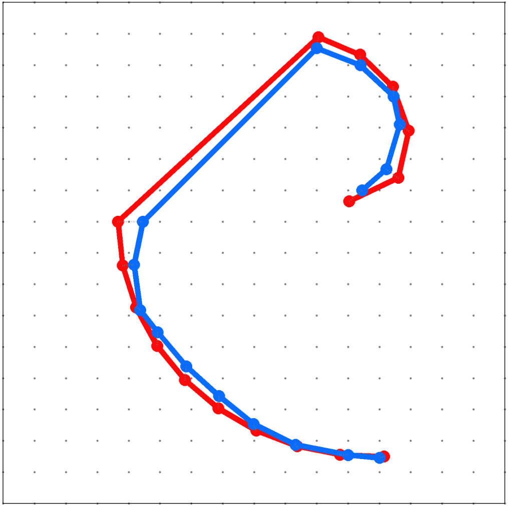
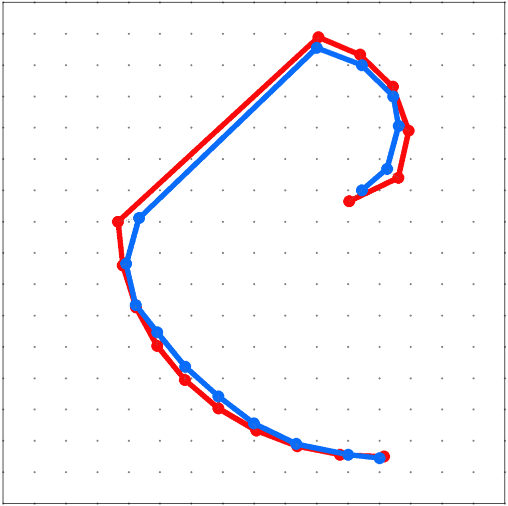


Bregman



Bregman



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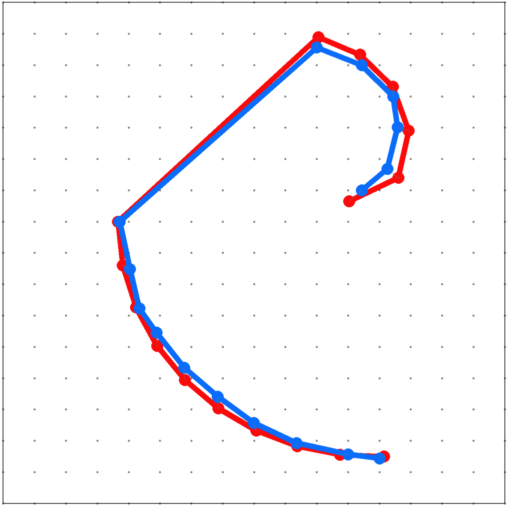

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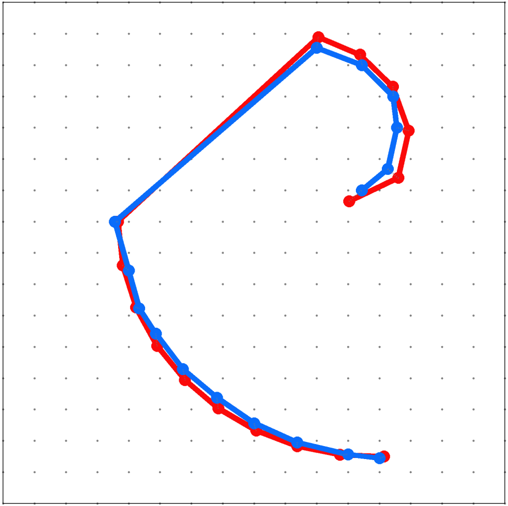
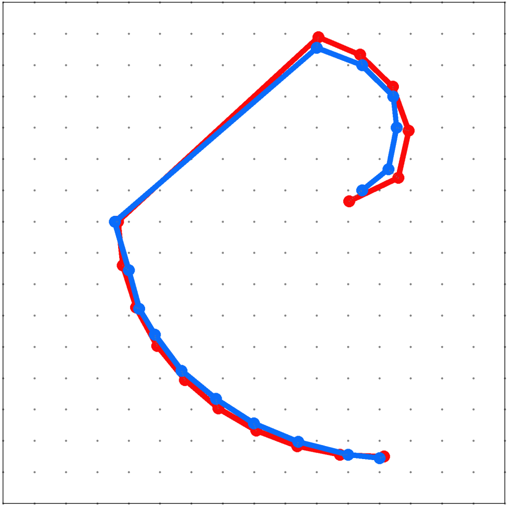
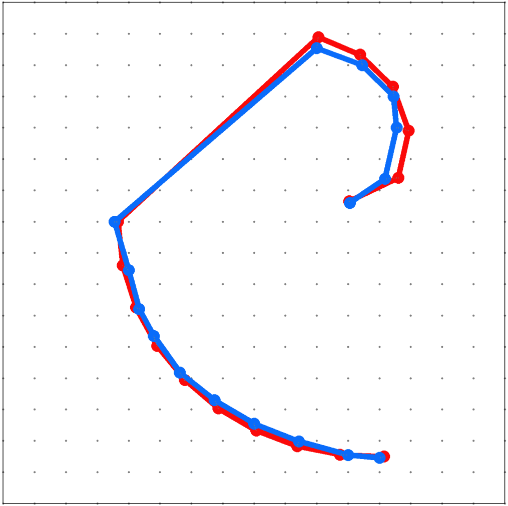


Bregman

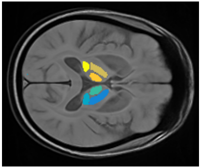


Bregman

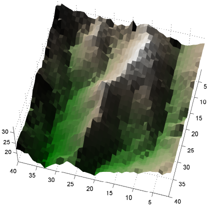


Conclusion

- **Finite label spaces – labelling/segmentation**
  - for *labelling* problems
  - convex relaxation in the function space



- **Infinite label spaces – manifolds**
  - for problems with values in $\mathbb{R}^n, \mathcal{S}^n, SO(3), \ldots$
  - also for more general convex regularisers
  - similar properties as in real-valued case: contrast loss, jump preservation, Bregman, ...
  - source code available



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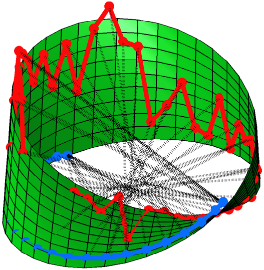
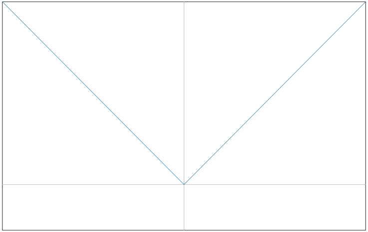
Nonconvex data terms

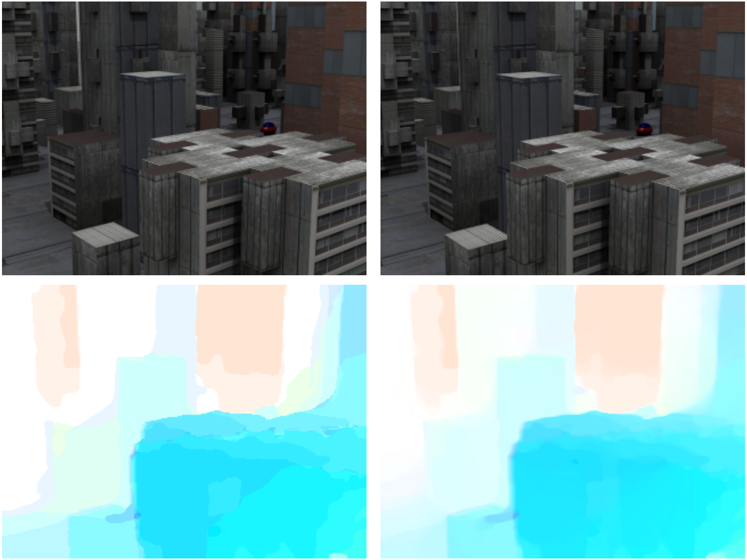
- *negative* L^2–TV (nonconvex)

$$\min_{u:\Omega\to\mathcal{M}} -D(u) + \lambda TV_{\mathcal{M}}(u)$$





Applications – optical flow



J. Lellmann, E. Strekalovskiy, S. Koetter, D. Cremers, ICCV 2013