Optimality Bounds for a Variational Relaxation of the Image Partitioning Problem

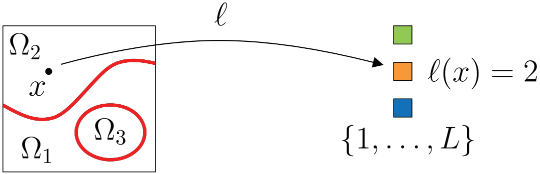
Jan Lellmann, Frank Lenzen, Christoph Schnörr

Image and Pattern Analysis Group
Universität Heidelberg

EMMCVPR 2011
St. Petersburg, July 25, 2011

Motivation – Problem

- Labeling problem:

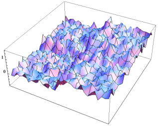
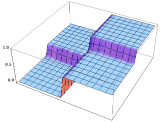


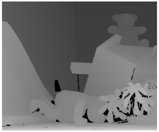
- Partition image domain Ω into L regions
- *Discrete* decision at each point in *continuous* domain Ω
- Variational Approach:

$$\min_{\ell} \underbrace{\int_{\Omega} s(\ell(x), x) dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

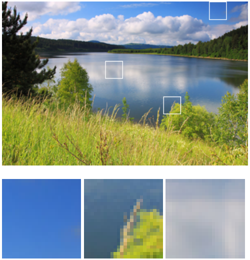
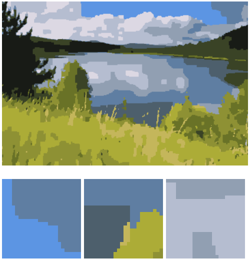
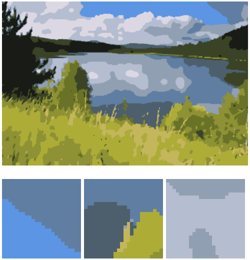
Motivation – Multiclass Labeling

- **Applications:** Denoising, segmentation, 3D reconstruction, depth from stereo, inpainting, photo montage, optical flow,...

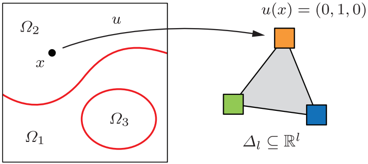


- Spatially continuous formulation avoids metrication artifacts:

Model – Multi-Class Labeling

- Multi-class relaxation: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]



- Embed labels into $\mathbb{R}^L$ as $\mathcal{E} := \{e^1, \ldots, e^L\}$, relax to the unit simplex:

$$\Delta_L := \{x \in \mathbb{R}^L | x \geqslant 0, \sum_i x_i = 1\} = \operatorname{conv} \mathcal{E},$$

$$\min_{u \in \mathrm{BV}(\Omega, \Delta_L)} f(u), \quad f(u) := \int_\Omega \langle u(x), s(x) \rangle dx + \int_\Omega \Psi(Du)$$

- Advantages: No explicit parametrization, rotation invariance, *convex*

Model – Rounding

- *Fractional* solutions may occur:

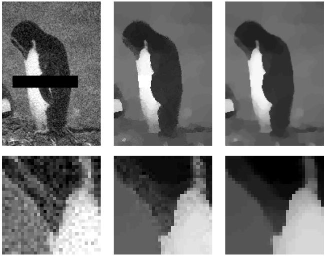
- **Goal:** Find *rounding scheme* $u^* \mapsto \bar{u}^* : \Omega \to \{e^1, \ldots, e^L\}$ such that

$$f(\underbrace{\bar{u}^*}_{\text{rounded relaxed solution}}) \leqslant (1+\delta) f(\underbrace{u^*_{\mathcal{E}}}_{\text{best integral solution}}).$$

for some $\delta \geqslant 0$.

Model – Rounding

- *A posteriori* approaches:
  [BaeYuanTai10, LellmannSchnörr10]
  - Require primal and/or dual solution
  - Depend strongly on specific input data



- Existing *a priori* bounds:
  - Discretized setting [Dahlhaus et al. 94, KleinbergTardos 99, Boykov et al. 01, KomodakisTziritas 07]
  - Continuous two-class case [Strang 83, ChanEsedogluNikolova 06, Zach et al. 09, Olsson et al. 09]
- **This talk:** *Spatially continuous*, *multi-class*, *a priori* bounds

Rounding – Two-Class Case

- *Two*-class case: Find region $C \subseteq \Omega$ minimizing [ChanVese01]

$$\min_{C} \int_{C} (I - c_1)^2 dx + \int_{\Omega \setminus C} (I - c_2)^2 dx + \lambda \mathcal{H}^{d-1}(\partial C)$$

- Reformulate using *indicator function* of C and relax:

$$\min_{u \in \mathrm{BV}(\Omega,[0,1])} \int_{\Omega} u(I - c_1)^2 + (1 - u)(I - c_2)^2 dx + \lambda TV(u)$$

Rounding – Two-Class Case

- Two-class case: Generalized *coarea formula*

$$f(u) \;=\; \int_0^1 f(\bar{u}_\alpha)d\alpha\,, \quad \bar{u}_\alpha := \begin{cases} 1, & u(x) > \alpha, \\ 0, & u(x) \leqslant \alpha. \end{cases}$$

- Also: Choquet integral, Lovász extension, levelable function,...
- Consequence: Global *integral* minimizer for a.e. α,

$$f(\bar{u}^*_\alpha) \;=\; f(u^*).$$

- This talk: *multi-class* generalization.

Rounding – Multi-Class Case

- *Probabilistic interpretation:*

$$f(u) \;=\; \int_0^1 f(\bar{u}_\alpha)d\alpha = \mathbb{E}_\alpha f(\bar{u}_\alpha)\,.$$

- Rounding step $u \mapsto \bar{u}_\alpha$ does not increase f *in the mean*
- Multi-class generalization (*approximate* generalized coarea formula):

$$(1+\delta)f(u) \;\geqslant\; \int_\Gamma f(\bar{u}_\gamma)d\mu(\gamma) = \mathbb{E}_\gamma f(\bar{u}_\gamma)$$

- Need to define:
  - *parameter space* Γ
  - parametrized *rounding method* $u \mapsto \bar{u}_\gamma$
  - *probability measure* μ on Γ
  - bound δ independent of input

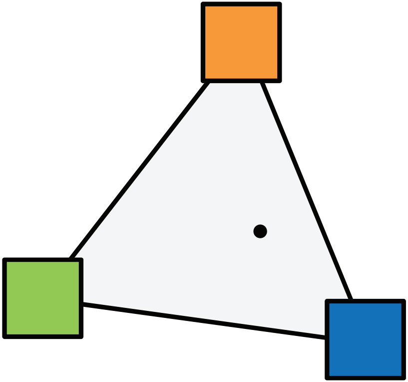
Optimality – Randomized Rounding

Algorithm 1 (Randomized Rounding in BV)

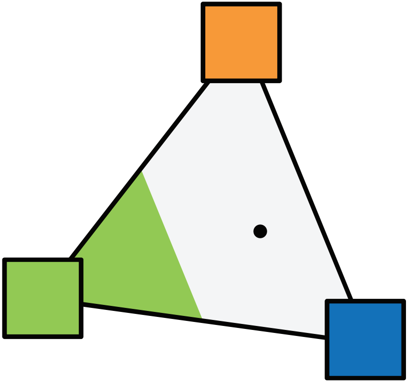
1. **Input:** $u^0 \in \mathrm{BV}(\Omega, \Delta_L)$
2. **For** $k = 1, 2, \ldots$
3. Sample $\gamma^k = (i^k, \alpha^k) \in \{1, \ldots, L\} \times [0, 1]$ uniformly
4. $u^k \leftarrow e^{i^k} 1_{\{u^{k-1}_{i^k} > \alpha^k\}} + u^{k-1} 1_{\{u^{k-1}_{i^k} \leqslant \alpha^k\}}$
5. **Output:** Limit $\bar{u}$ of (u^k)

- Parameter space: *sequences* $\gamma \in \Gamma := (\{1, \ldots, L\} \times [0, 1])^{\mathbb{N}}$

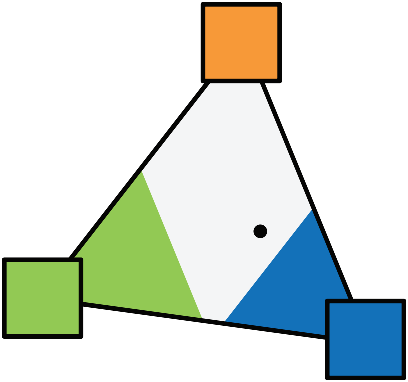
Optimality – Example



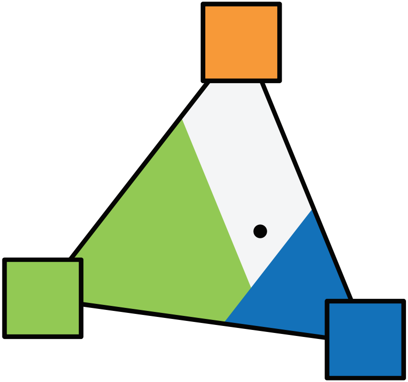
Optimality – Example



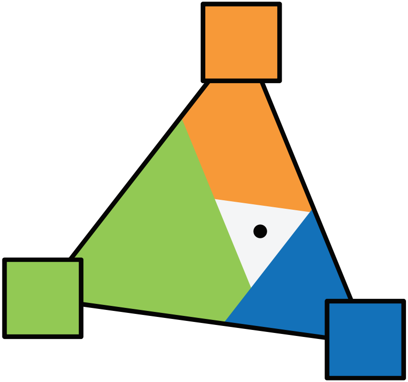
Optimality – Example



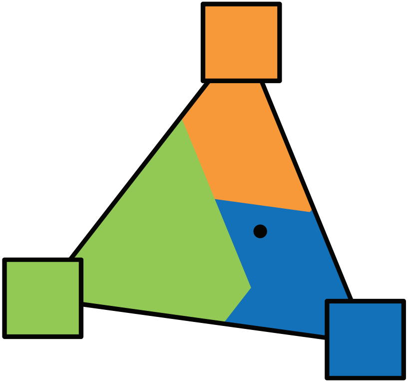
Optimality – Example



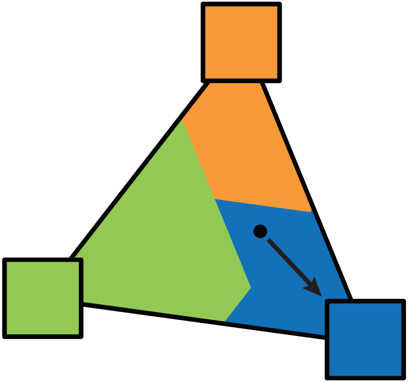
Optimality – Example



Optimality – Example



Optimality – Example



Optimality – Main Result

Theorem (Optimality)

Let $u \in \mathrm{BV}(\Omega, \Delta_L)$, $s \in L^1(\Omega)^L$, $s \geqslant 0$, $\Psi : \mathbb{R}^{d \times l} \to \mathbb{R}_{\geqslant 0}$ *positively homogeneous, convex and continuous, and* $\lambda_l > 0, \lambda_u < \infty$ *such that*

$$\begin{aligned} \Psi(z = (z^1, \ldots, z^L)) &\geqslant \lambda_l \frac{1}{2} \sum_{i=1}^{L} \|z^i\|_2 \quad \forall z \in \mathbb{R}^{d \times L}, \sum_{i=1}^{L} z^i = 0, \\ \Psi(\nu(e^i - e^j)^\top) &\leqslant \lambda_u \quad \forall i, j \in \{1, \ldots, L\}, \nu \in \mathbb{R}^d, \|\nu\|_2 = 1. \end{aligned}$$

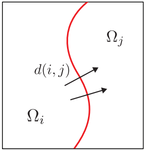
Then

$$\mathbb{E} f(\bar{u}) \leqslant 2 \frac{\lambda_u}{\lambda_l} f(u) \qquad \text{and} \qquad \mathbb{E} f(\bar{u}^*) \leqslant 2 \frac{\lambda_u}{\lambda_l} f(u_{\mathcal{E}}^*).$$

- Provides "approximate" generalized coarea formula

Optimality – Metric Labeling

- Specially designed Ψ: *Local envelope* relaxation for metric interaction potentials [Chambolle et al. 08, LellmannSchnörr 10]



- For a given metric d,

$$\mathbb{E}f(\bar{u}^*) \leqslant 2\frac{\max_{i\neq j} d(i,j)}{\min_{i\neq j} d(i,j)} f(u_{\mathcal{E}}^*).$$

- Compatible with bounds for finite-dimensional multiway cut, α-expansion, LP relaxation
- Formulated in BV, independent of discretization

Summary

- **Motivation:**
  - Convex relaxation of multiclass image labeling
  - Solutions may be fractional!
  - *Goal: A priori* bounds for *rounded* solution
- **Rounding Strategy:**
  - *Probabilistic* approach
  - Motivated by LP framework
- **Bounds:**
  - *A priori* bound
  - Provides approximate generalized coarea formula
  - Compatible with finite-dimensional results
  - Fully *spatially continuous* BV framework

Optimality Bounds for a Variational Relaxation of the Image Partitioning Problem

Jan Lellmann, Frank Lenzen, Christoph Schnörr

Image and Pattern Analysis Group
Universität Heidelberg

EMMCVPR 2011
St. Petersburg, July 25, 2011

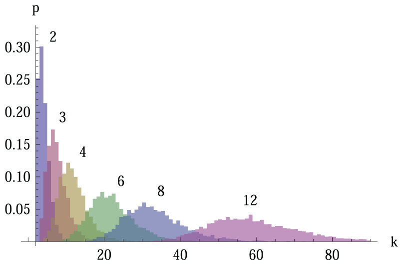
Optimality – Termination

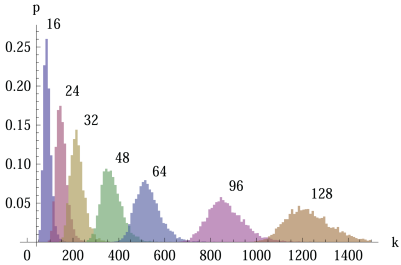
Theorem (Termination)

Let $u \in \mathrm{BV}(\Omega, \Delta_L)$. Then (almost surely) Alg. 1 generates a sequence that becomes stationary in some $\bar{u} \in \mathrm{BV}(\Omega, \mathcal{E})$.

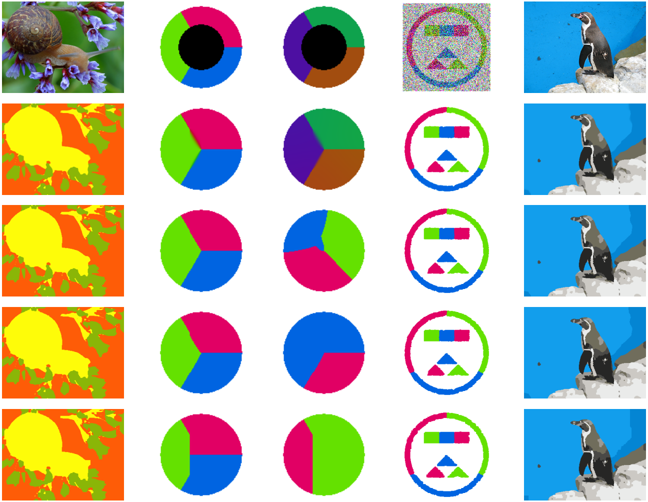
- Result is in BV
- Independent of data term

Experiments – Iterations

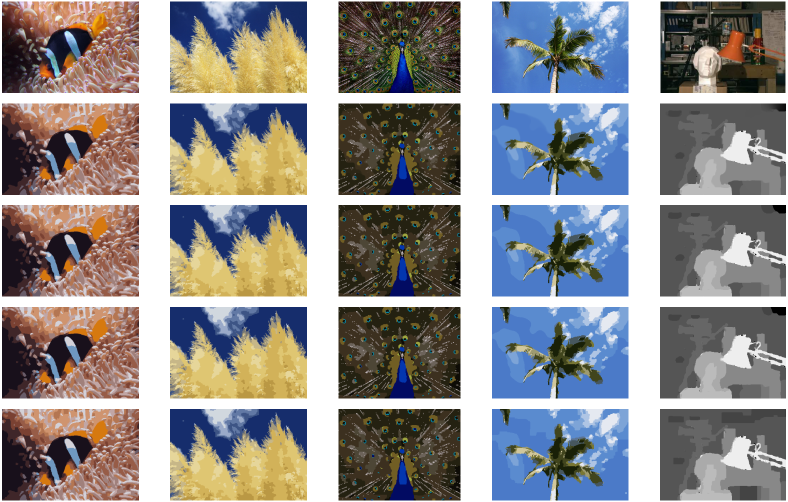




Experiments I



Experiments II



Experiments – Results

problem	1	2	3	4	5	6	7	8	9	10
# points	76800	14400	14400	129240	76800	86400	86400	76800	86400	110592
# labels	3	4	4	4	8	12	12	12	12	16
mean # iter.	7.27	7.9	8.05	10.79	31.85	49.1	49.4	49.4	49.7	66.1
a priori ε	1.	1.	1.	1.	1.	1.	1.	1.	1.	2.6332
a posteriori										
- first-max	0.0007	0.0231	0.2360	0.0030	0.0099	0.0102	0.0090	0.0101	0.0183	0.0209
- prob. best	0.0010	0.0314	0.1073	0.0045	0.0177	0.0195	0.0174	0.0219	0.0309	0.0487
- prob. mean	0.0007	0.0231	0.0547	0.0029	0.0138	0.0152	0.0134	0.0155	0.0247	0.0292