Fast Numerical Methods for Continuous Multiclass Labeling

J. Lellmann, C. Schnörr

Image and Pattern Analysis Group
University of Heidelberg

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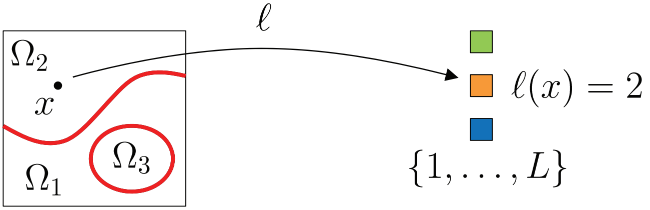
Problem

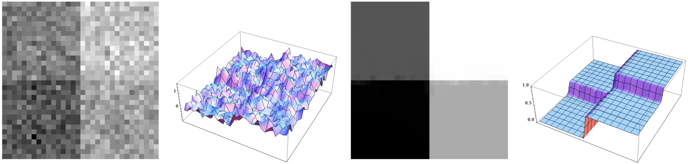
- Multi-Class Labeling



- Partition into L regions
- Using *data term (local)* and *regularizer (prior/smoothness)*

Applications

- Denoising/color segmentation

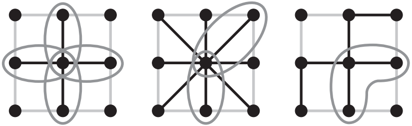


- Image segmentation
- Stereo matching

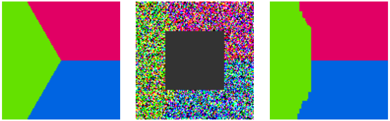
- Inpainting, photo montage, etc...

Motivation

- Pairwise Markov Random Fields [Geman84]: variants of belief propagation, graph cuts with expansion/swap moves [Boykov01]



- Difficult to parallelize, grid bias:

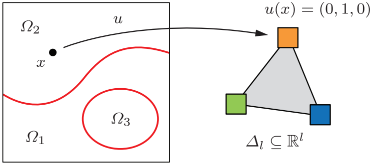


- $\Rightarrow$ Continuous approach [Strang81,Nikolova04,Zach08,Chambolle08,Lellmann09]

$$\min_{\ell:\Omega\to\{1,\ldots,L\}} \underbrace{\int_\Omega s(\ell(x),x)dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

Approach – Relaxation

- Embed labels into $\mathbb{R}^L$ as $\{e^1, \ldots, e^L\}$ (unordered!) and relax to unit simplex:



- Continuous convex formulation:

$$\min_{u \in \mathrm{BV}(\Omega, \Delta_L)} f(u)\,, \qquad f(u) = \int_\Omega \langle u(x), s(x) \rangle dx + TV(Au)$$

- *Linear* data term, *convex* problem, but *nonsmooth*
- Minimizer exists in BV, not necessarily unique (linear growth)

Regularizer

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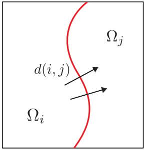
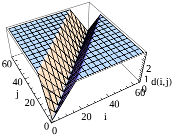
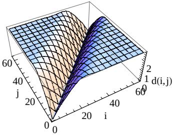
- $TV(u)$ measures length of interface (once!)
- Better: weight by *interaction potential* (metric) $d(i,j)$



- Embed *Euclidean* $d(i,j) = \|a^i - a^j\|$ using $TV(Au)$
- Approximate non-Euclidean potentials using SDP





Optimization – Problem

- Dual formulation of TV

$$\mathrm{TV}(Au) = \int_\Omega \|D(Au)\| = \sup_{v \in C_c^1(\Omega)^{d \times L}, \|v(x)\| \leqslant 1 \text{ a.e.}} \int_\Omega \langle Au, \mathrm{Div}\, v \rangle dx$$

- (Bi-)Linear Saddle-Point Problem:

$$\min_{u \in \mathcal{C}} \max_{v \in \mathcal{D}} \{ \langle u, s \rangle + \langle Lu, v \rangle \}$$

- Convex but nonsmooth with constraints
- Approaches:
  - Explicit smoothing: critical due to properties of solution
  - Interior Point (SOCP), Repeated Binary Fusion [Trobin08,Olsson09]
  - Here: First order primal/dual methods: small memory footprint, easy to parallelize, provide certificate

Optimization – Approaches

- Saddle-Point Problem:

$$\min_{u \in \mathcal{C}} \max_{v \in \mathcal{D}} \left\{ \langle u, s \rangle + \langle Lu, v \rangle \right\}$$

- Fast Primal Dual/Modified Popov [Popov80,Pock09]
  - Combine alternating gradient-projection steps on u and v
  - Step size restriction
- Nesterov [Nesterov04,Lellmann09]
  - Controlled smoothing + smooth first-order multistep method
  - ε-optimal solution in $O(1/\varepsilon)$, explicit bounds

Optimization – Douglas-Rachford Splitting

- Split objective:

$$\min_{u,w} \underbrace{\delta_{Lu=w}(u,w)}_{f_1} + \underbrace{\langle u, s\rangle + \delta_{\mathcal{C}}(u) + \sigma_{\mathcal{D}}(w-b)}_{f_2}.$$

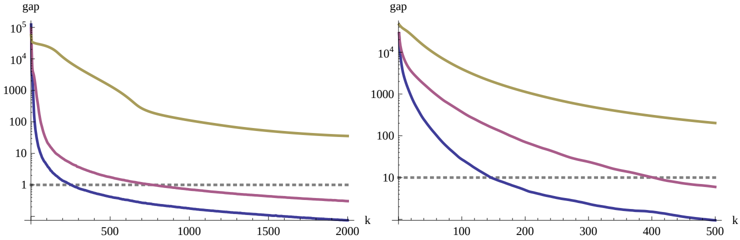
- Douglas-Rachford Splitting: Iterate proximal steps to find fixpoint of

$$\bar{z}^{k+1} \quad \in \quad (J_{\tau\partial f_1}(2J_{\tau\partial f_2} - I) + (I - J_{\tau\partial f_2}))(\bar{z}^k),$$

- Requires only L, $L^\top$, $\Pi_{\mathcal{C}}$, $\Pi_{\mathcal{D}}$, DCT.
- Globally convergent under mild assumptions for any step size τ [e.g. Eckstein1989].
- Related to Alternating Split Bregman, Alternating Direct Method of Multipliers [Gabay83,Setzer09].

Experiments – Performance

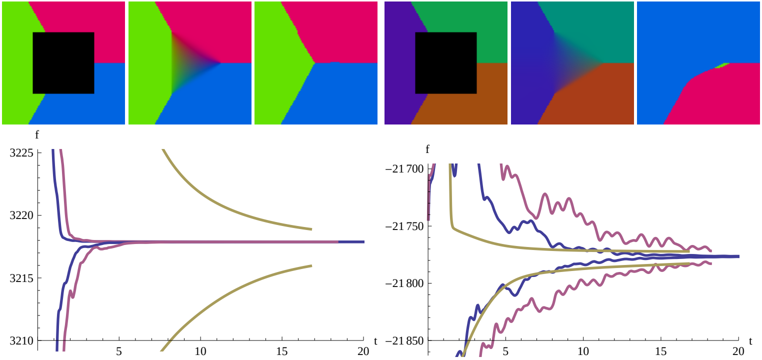
- Faster than FPD and Nesterov:



- Relatively faster if projections expensive relative to DCT

Experiments – Oscillation

- Difficult relaxed problem:



Experiments – Relaxation

- Almost binary relaxed solution:

- 90.6% of pixels closer than 0.05 to “hard” label.

Experiments – Thresholding

- Thresholding does not preserve optimality as in two-class case! But:

$$f(u_{\text{thresh}}) - f(u^*_{\text{discr}}) \leqslant f(u_{\text{thresh}}) - f_d(v)$$

- Usually bound of $1\% - 6\%$ for *combinatorial* problem
- Basic “maximal component” thresholding suboptimal:

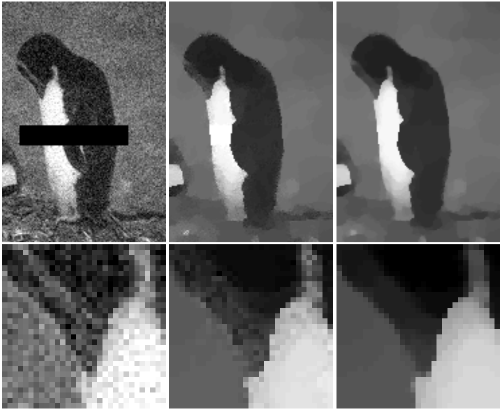
$$d(i,j) = |i-j|, \quad A = (0\,1\,2), \quad u(x) = (1/3+\delta, 1/3, 1/3-\delta)$$

- Better:

$$\ell(x) = \arg\min_i \|A(u - e^i)\|$$

Experiments – Thresholding

- Proper thresholding improves result



- Relative gap 2.78% vs. 15.78%

Summary

- Formulate *multi-class* labeling as *continuous convex* problem.
- Variational formulation, no grid bias
- Nonsmooth convex problem $\rightarrow$ bilinear saddle-point

- Douglas-Rachford splitting
  - Robust, easy to parallelize, $1/3$ iterations of FPD
- Proper thresholding $\Rightarrow$ good discrete solutions

Extensions – General Metrics

- How to improve tightness of relaxation?
- Construct local convex envelope [Chambolle08] $(d(i,j) = \sigma(|i-j|))$
- Generally, for support function $\Psi = \sigma_{\mathcal{D}_{\text{loc}}}$

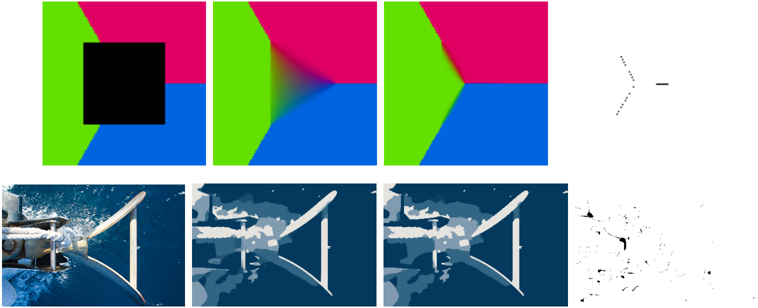
$$
\begin{aligned}
& J(u) = \int_\Omega \Psi(Du) \\
& \Rightarrow J(e^i(1-u') + e^j u') = \Psi(y \otimes (e^j - e^i))\, \mathrm{TV}(u'), \|y\| = 1
\end{aligned}
$$

- Generalize to arbitrary metrics $d(i,j)$ [Lellmann10]

$$
\mathcal{D}_{\text{loc}} = \{(v^1, \ldots, v^L) \in \mathbb{R}^{d \times L} | \|v^i - v^j\| \leqslant d(i,j), \sum_k v^k = 0\}
$$

Extensions – General Metrics

- Improved tightness



- Some drawbacks: inexact projections, no primal objective

Extensions – General Functionals

- Goal: minimize

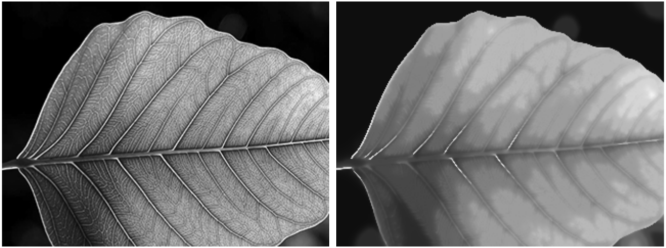
$$J(u') = \int_\Omega f(x, u'(x), \nabla u'(x))$$

- Represent non-smooth u as hypograph $1_{t \leq u'(x)}$, relax to $[0,1]$

$$\min_{u \in \mathrm{BV}(\Omega \times \mathbb{R}, [0,1])} \sup_{v \in \mathcal{D}} \int_{\Omega \times \mathbb{R}} \langle v, Du \rangle$$

$$\mathcal{D} = \{(v^x, v^t) | v^t(x,t) \geqslant f^*(x, t, v^x(x,t)) \; \forall (x,t) \in \Omega \times \mathbb{R}\}$$

- Full Mumford-Shah [Chambolle08,Pock09]



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