# Convexity and Non-Convexity in Partitioning and Interpolation Problems

Jan Lellmann

CIA/DAMTP, University of Cambridge

Joint work with: C. Schnörr, F. Lenzen (IPA/HCI, University of Heidelberg),
F. Widmann, B. Lellmann (Imperial College, London),
C. Schönlieb (DAMTP, University of Cambridge)
J.-M. Morel (ENS Cachan)
Acknowledgments: A. Bertozzi and A. Chen

IPAM, Feb 2013

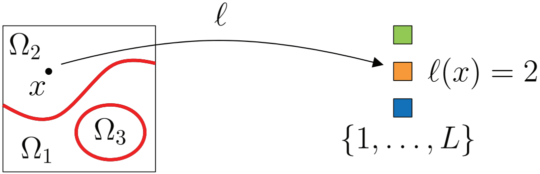
# Overview

1. Optimality bounds for convex labelling

2. Non-convex surface interpolation

# Motivation – Problem

- Labeling problem:

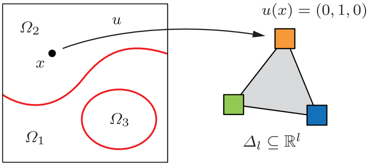



- Partition image domain $\Omega$ into $L$ regions
- *Discrete* decision at each point in *continuous* domain $\Omega$
- Variational Approach:

$$\min_{\ell} \underbrace{\int_{\Omega} s(\ell(x), x) dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

# Model – Multi-Class Labeling

- Multi-class relaxation: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]




- Embed labels into $\mathbb{R}^L$ as $\mathcal{E} := \{e^1, \ldots, e^L\}$, relax integrality constraint to the unit simplex:

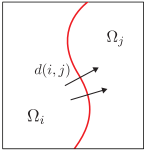
$$\Delta_L := \{x \in \mathbb{R}^L | x \geqslant 0, \sum_i x_i = 1\} = \operatorname{conv} \mathcal{E},$$

$$\min_{u \in \mathrm{BV}(\Omega, \Delta_L)} f(u), \quad f(u) := \int_\Omega \langle u(x), s(x) \rangle dx + \int_\Omega \Psi(Du)$$

- Advantages: No explicit parametrization, rotation invariance, *convex*

# Model – Envelope Relaxation

- $J(\ell)$: Weight boundary length by *interaction potential* $d(i,j)$




- $\psi$ implicitly defined for given $d$ [Chambolle/Cremers/Pock '08,Lellmann/Schnoerr '10] $\longrightarrow$ *metric labeling*.

$$
\begin{aligned}
J(u) &:= \sup_{v \in \mathcal{D}} \int_\Omega \langle Du, v\rangle = \int_\Omega \underbrace{\sigma_{\mathcal{D}_{\text{loc}}}(Du)}_{\Psi(Du)}, \\
\mathcal{D} &:= \{v \in (C_c^\infty)^{d\times L} | v(x) \in \mathcal{D}_{\text{loc}}\ \forall x \in \Omega\}, \\
\mathcal{D}_{\text{loc}} &:= \{(v^1, \ldots, v^L) \in \mathbb{R}^{d\times L} | \|v^i - v^j\| \leqslant d(i,j)\ \forall i,j\}.
\end{aligned}
$$

- Primal formulation, connections to local polytope [Zach/Häne/Pollefeys '12]
- Quantized but versatile, on-/offline methods to reduce constraints

[Chambolle/Cremers/Pock '12, Lellmann et al. '13]

# Model – Rounding

- *Fractional* solutions may occur:

- **Goal:** Find *rounding scheme* $u^* \mapsto \bar{u}^* : \Omega \to \{e^1, \ldots, e^L\}$ such that

$$f(\underbrace{\bar{u}^*}_{\text{rounded relaxed solution}}) \leqslant C f(\underbrace{u^*_{\mathcal{E}}}_{\text{best integral solution}}).$$

for some $C \geqslant 1$.

# Rounding – Generalized Coarea Formula

- Two-class case: Generalized *coarea formula* [Strang '83, Chan/Esedoglu/Nikolova '06, Zach et al. 09, Olsson et al. 09]

$$f(u) \quad = \quad \int_0^1 f(\bar{u}_\gamma) d\gamma \,, \quad \bar{u}_\gamma := \begin{cases} e^1, & u_1(x) > \gamma, \\ e^2, & u_1(x) \leqslant \gamma. \end{cases}$$

- Also: *Choquet integral, Lovász extension, levelable function*,...

# Rounding – Multi-Class Case

- *Probabilistic interpretation:*

$$f(u) \quad = \quad \int_0^1 f(\bar{u}_\gamma) d\gamma = \mathbb{E}_\gamma f(\bar{u}_\gamma)\,.$$

- Rounding step $u \mapsto \bar{u}_\gamma$ does not increase $f$ *in the expectation*
- Consequence: global *integral* minimizer for a.e. $\gamma$! [Chan/Esedoglu/Nikolova '06]

# Rounding – Approximate Coarea Formula

- Multi-class generalization (*approximate* generalized coarea formula):

$$C f(u) \;\geqslant\; \int_{\Gamma} f(\bar{u}_{\gamma}) d\mu(\gamma) = \mathbb{E}_{\gamma} f(\bar{u}_{\gamma})$$

- Rounding step $u \mapsto \bar{u}_{\gamma}$ does not increase $f$ too much *in the expectation*
- No bounds for *individual* $\gamma$!

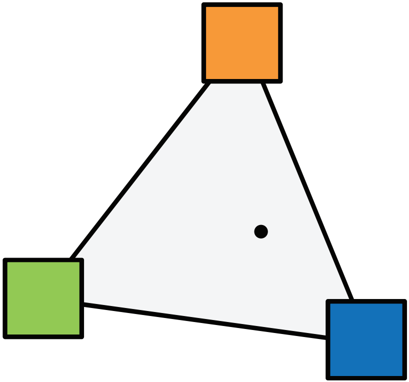
# Rounding – Multi-Class Case

- Multi-class generalization (*approximate* generalized coarea formula):

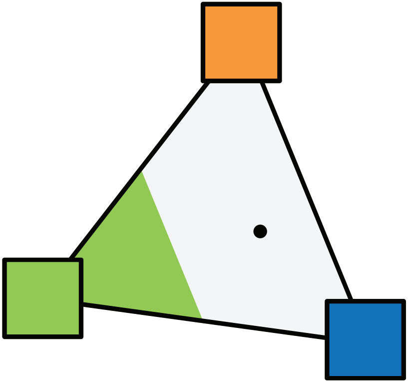
$$C f(u) \;\geqslant\; \int_\Gamma f(\bar{u}_\gamma) d\mu(\gamma) = \mathbb{E}_\gamma f(\bar{u}_\gamma)$$

- Need to define:
  - *parameter space* $\Gamma$
  - parametrized *rounding method* $u \mapsto \bar{u}_\gamma$
  - *probability measure* $\mu$ on $\Gamma$
  - bound $C$ independent of input

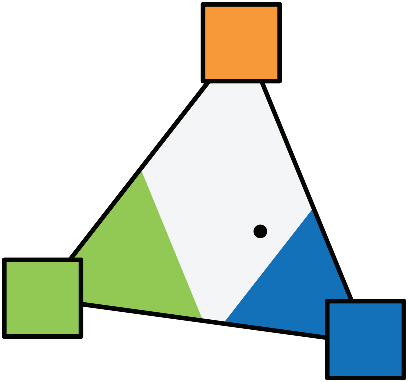
# Optimality – Example



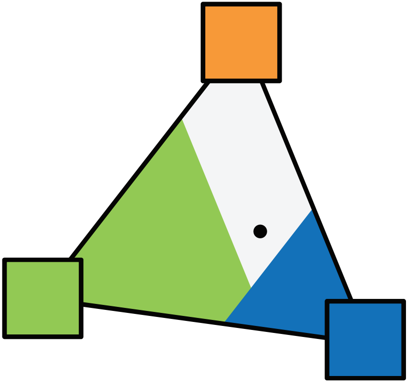
# Optimality – Example



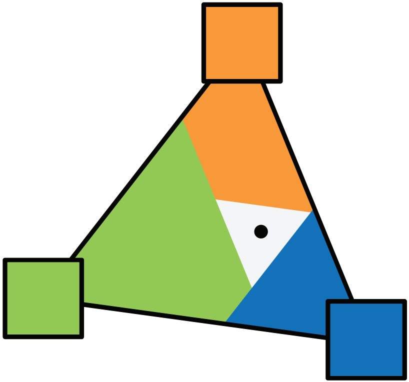
# Optimality – Example



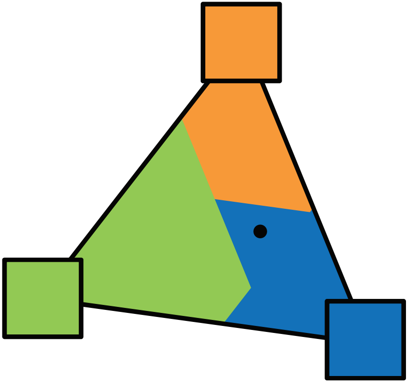
# Optimality – Example



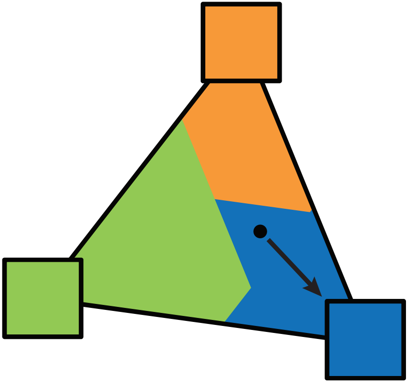
# Optimality – Example



# Optimality – Example



# Optimality – Example



# Optimality – Randomized Rounding

**Algorithm 1 (Randomized Rounding in BV)**

1. **Input:** $u^0 \in \mathrm{BV}(\Omega, \Delta_L)$
2. **For** $k = 1, 2, \ldots$
3. Sample $\gamma^k = (i^k, \alpha^k) \in \{1, \ldots, L\} \times [0, 1]$ uniformly
4. $u^k \leftarrow \begin{cases} e^{i^k}, & u^{k-1}_{i^k} > \alpha^k, \\ u^{k-1}, & u^{k-1}_{i^k} \leqslant \alpha^k. \end{cases}$
5. **Output:** Limit $\bar{u}$ of $(u^k)$

- Parameter space: *sequences* $\gamma \in \Gamma := (\{1, \ldots, L\} \times [0, 1])^{\mathbb{N}}$

**Theorem (Termination)**

*Let* $u \in \mathrm{BV}(\Omega, \Delta_L)$. *Then (almost surely) Alg. 1 generates a sequence that becomes stationary in some* $\bar{u} \in \mathrm{BV}(\Omega, \mathcal{E})$.

- Result is in BV
- Independent of data term

# Randomized Rounding – Motivation

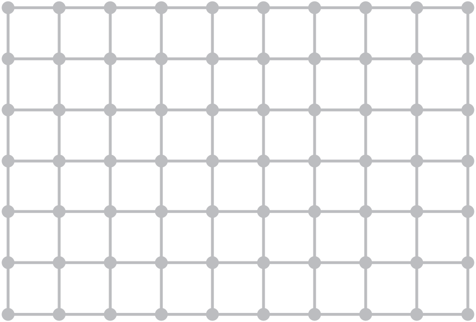
- In finite-dimensional setting: Can show for uniform metric [Kleinberg/Tardos '02]:

$$\mathbb{E}_\gamma f(\bar{u}_\gamma) \leqslant 2 f(u).$$

- Similar bounds:
  - multiway cut [Dahlhaus et al. '94]
  - $\alpha$-expansion [Boykov et al. '01]
  - LP relaxation [Komodakis/Tziritas '07]

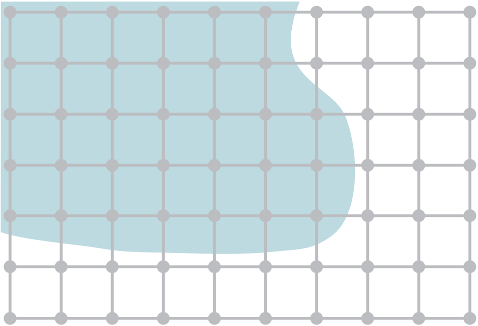
# Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



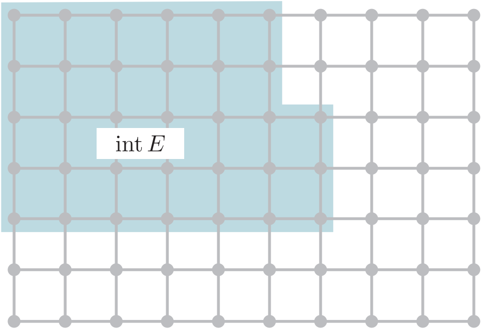
# Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



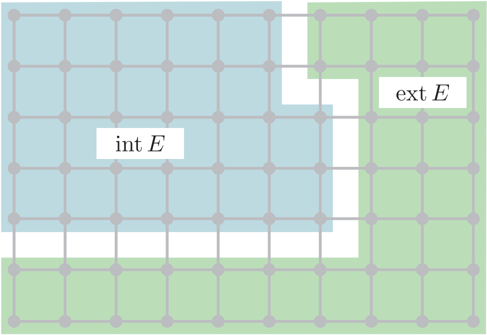
# Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



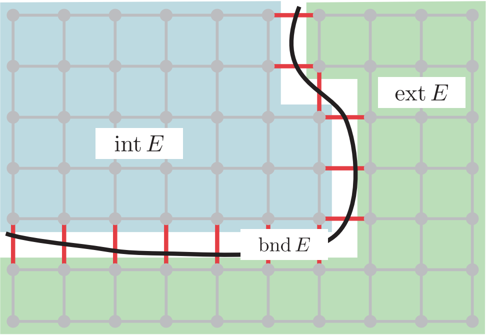
# Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



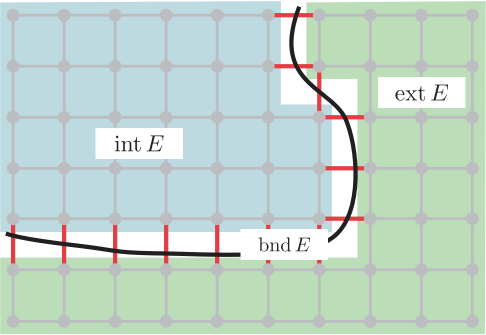
# Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



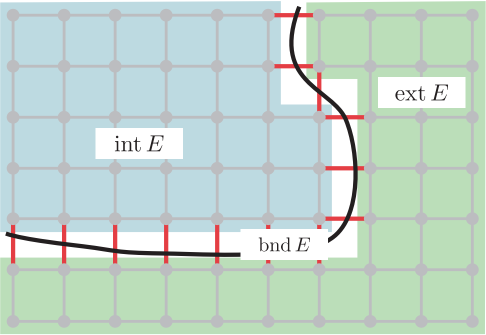
# Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$




$$\mathbb{R}^n : \quad J(u) \leqslant J|_{\text{int}\, E}(u) + J|_{\text{ext}\, E}(u) + c\,\text{Per}(E)$$

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$




$$\mathbb{R}^n : \quad J(u) \leqslant J|_{\text{int}\, E}(u) + J|_{\text{ext}\, E}(u) + c\,\text{Per}(E)$$

$$\text{BV}(\Omega) : |\psi(Du)|(\Omega) \leqslant |\psi(Du)|(E^1) + |\psi(Du)|(E^0) + c\,\text{Per}(E),$$

$$(E)^t := \{x \in \Omega | \lim_{\rho \searrow 0} \frac{|\mathcal{B}_\rho(x) \cap E|}{|\mathcal{B}_\rho(x)|} = t\}, \quad t \in [0,1].$$

# Optimality – Main Result

**Theorem (Optimality** [Lellmann/Lenzen/Schnoerr '11]**)**

*Let* $u \in \mathrm{BV}(\Omega, \Delta_L)$, $s \in L^\infty(\Omega)^L$, $s \geqslant 0$, $d$ *metric. Then*

$$\mathbb{E} f(\bar{u}) \leqslant 2 \frac{\max_{i \neq j} d(i,j)}{\min_{i \neq j} d(i,j)} f(u) \qquad \textit{and} \qquad \mathbb{E} f(\bar{u}^*) \leqslant 2 \frac{\max_{i \neq j} d(i,j)}{\min_{i \neq j} d(i,j)} f(u^*_{\mathcal{E}}).$$

- Provides "approximate" generalized coarea formula
- Compatible with bounds for finite-dimensional multiway cut, $\alpha$-expansion, LP relaxation
- Formulated in BV, independent of discretization, true *a priori* bound independent of problem instance

# Discretization – Issues

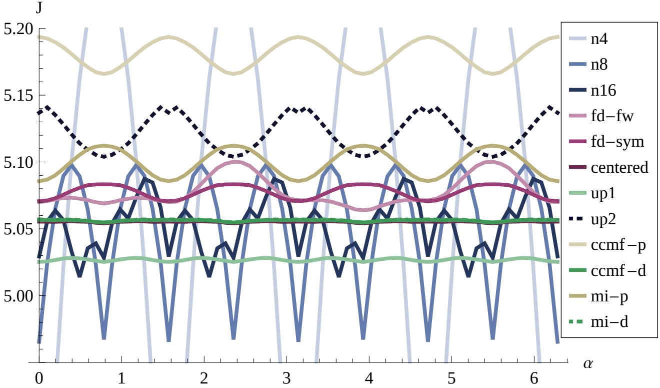
- Discretization affects *geometry* and *integrality*

- Questions:
  - Consistency
  - Isotropy
  - Convergence
- *Not* only asymptotic!

# Discretization – Finite Differences

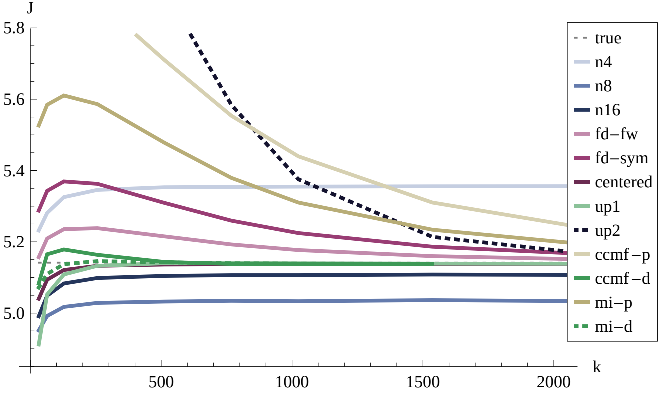
- pairwise LP (`n4`, `n8`, `n16`) [Boykov/Kolmogorov '03]
- forward differences (`fd-fw`), symmetrized (`fd-sym`) – often used
- staggered grid (`center`)
- upwind first-/second-order (`up1`,`up2`) [Rickett/Fomel '11, Chambolle/Levine/Lucier '11]
- CCMF (`ccmf-d`), dual (`ccmf-p`) [Couprie et al. '11], mimetic (`mi-d`), dual (`mi-p`) [Hyman/Shashkov '97, Yuan/Schnörr/Steidl '09] – dual constraints non-separable
- Also: finite elements – need adaptive mesh [Negri '99], dual methods

- Isotropy


J
5.20
5.15
5.10
5.05
5.00
0
1
2
3
4
5
6
α
n4
n8
n16
fd−fw
fd−sym
centered
up1
up2
ccmf−p
ccmf−d
mi−p
mi−d

# Discretization

- Consistence



# Discretization

- Minimizers for rotationally symmetric data term:

n4 (Wulff) | n8 | n16 | fd-fw | (fd-fw) | fd-sym | (fd-sym)

center | up1 | up2 | ccmf-p | ccmf-d | mi-p | mi-d

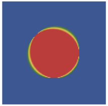
# Discretization

- Minimizers for rotationally symmetric data term:

n4 (Wulff) 100.00

n8 100.00

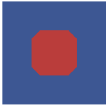
n16 100.00



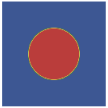
fd-fw 94.71

(fd-fw) 100.00

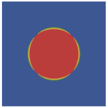
fd-sym 93.33



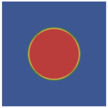
(fd-sym) 100.00



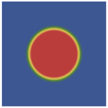
center 97.19



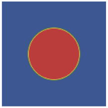
up1 96.44



up2 94.17

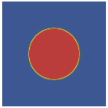


ccmf-p 89.21



ccmf-d 96.66

mi-p 92.02



mi-d 96.24

- Criterion: values closer than 0.05 to $\{0,1\}$
- Centered differences most precise
- Much less exaggerated on real-world data

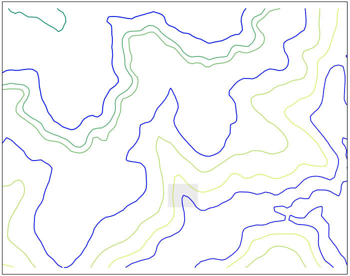
# Overview

1. Optimality bounds for convex labelling

2. Non-convex surface interpolation

# Surface interpolation

- *Input:* (parts of) contour lines
- *Output:* dense surface

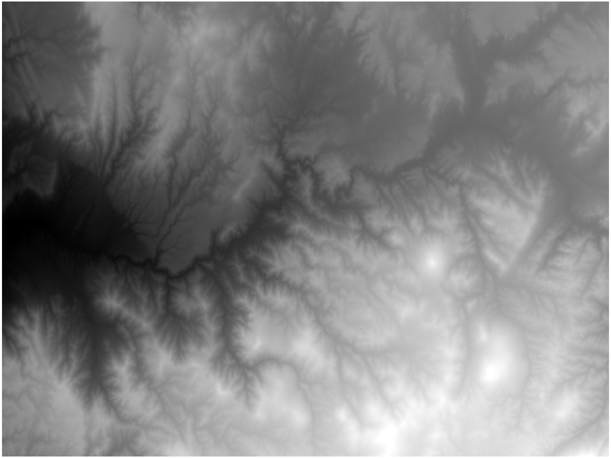


- Variational approach:

$$\min_{u:\Omega\to\mathbb{R}} R(u), \quad \text{s.t. } u(x) = u_0(x) \text{ for } x \in C.$$

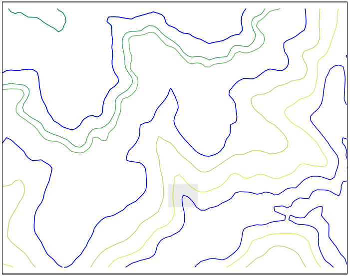
# Digital elevation map

- Standard regularizers highly unlikely to work



# Challenges – 1

- Challenges:
  - contour lines can have non-differentiabilities/high curvature, these are *features* and should be preserved!
  - data can be irregular/sparse – regularizer is much more important
  - discretization has large influence (also: boundary conditions)

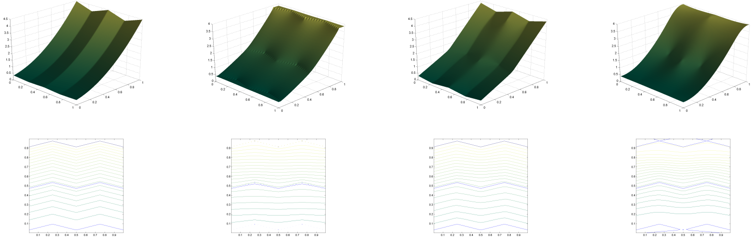


# Related work

- Goal: sparse data, no explicit parameterization, at least continuous, sharp ridges, function space domain
- Related methods:
  - Explicit parameterisation [Meyers et al. '92, Masnou/Morel '98, Hormann et al. '03, Meyer '11]
  - Geodesic distance transform [Soille '91]
  - Membrane, thin plate spline [Duchon '76] – smooth contours
  - Kriging [Matheron '71, Stein '99]
  - Anisotropic Diffusion [Desbrun '00], of normals [Tasdizen '02]
  - Total curvature [Elsey/Esedoglu '07]
  - Implicit representation [Ye et al. '10], higher-order TV [Lai/Tai/Chan '11]
  - Absolutely Minimizing Lipschitz Extension [Alvarez et al.'93, Caselles et al. '98] – $D^2 u(Du/|Du|, Du/|Du|) = 0$,
  - many more...

# Classic approaches

- Smoothed-out contours

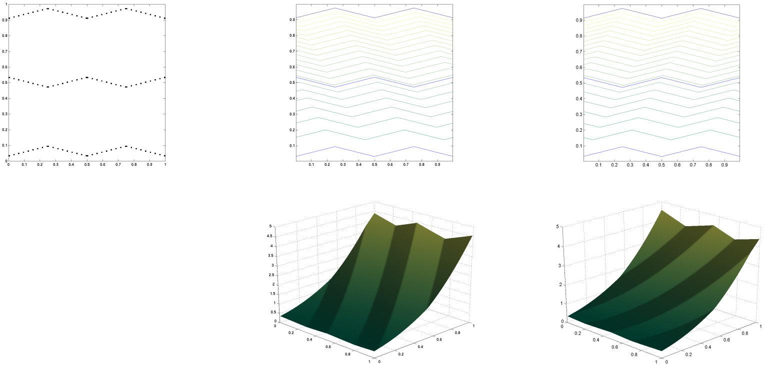


input | quadratic | $TV^2$ | $TV^3$

- convex?

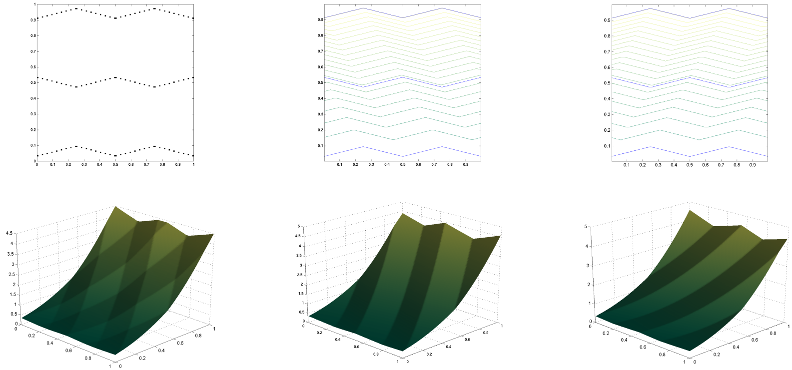
# Challenges – 2

- Is convex regularization enough?



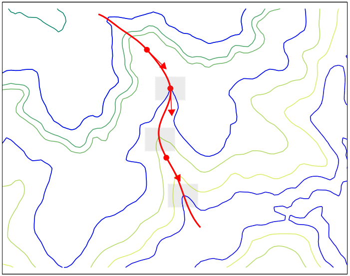
# Challenges – 2

- Is convex regularization enough?



# Auxiliary vectors

- Need to introduce non-convexity
- Contours are similar
- Smooth *across* level lines – vector field $v$ associates points on contours

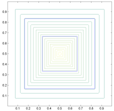


- Introduce anisotropy:

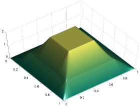
$$R_1^{(3)}(u) := \int_\Omega \|D^3 u(v,\cdot,\cdot)\|,\; R_2^{(3)}(u) := \int_\Omega \|D^3 u(v,v,\cdot)\|,\; R_3^{(3)}(u) := \int_\Omega \|D^3 u(v,v,v)\|,$$

$$R_1^{(2)} := \int_\Omega \|D^2 u(v,\cdot)\|,\; R_2^{(2)} := \int_\Omega \|D^2 u(v,v)\|$$

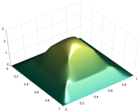
- Known $v$:



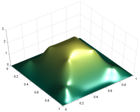
input



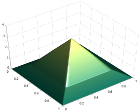
AMLE



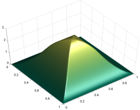
$\|D^2 u\|$



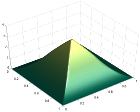
$\|D^3 u\|$



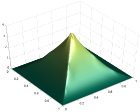
$\|D^2 u(v, \cdot)\|$, $|D^2 u(v, v)|$



$\|D^3 u(v, \cdot, \cdot)\|$



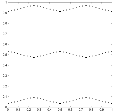
$\|D^3 u(v, v, \cdot)\|$



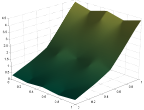
$|D^3 u(v, v, v)|$

- $v = Du/|Du|$ is *not* enough (AMLE)

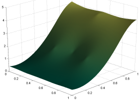
- Resolves ambiguities:



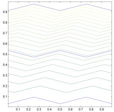
input

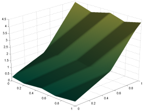


$\|D^2 u\|$

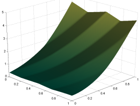


$\|D^3 u\|$

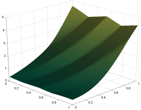




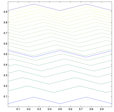
$\|D^2 u(v, \cdot)\|$

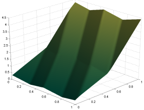


$\|D^3 u(v, \cdot, \cdot)\|$

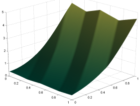


$\|D^3 u(v, v, \cdot)\|$

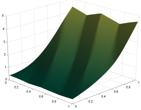




$\|D^2 u(v, \cdot)\|$



$\|D^3 u(v, \cdot, \cdot)\|$



$\|D^3 u(v, v, \cdot)\|$

# Finding $v$

- *Idea:* $v$ locally points into direction where *normal* to contour line changes least – assume contour lines are locally only translated:

$$v(x) \quad = \quad \arg \min_{w, \|w\|_2 = 1} \|K_\sigma * (D(Du/|Du|))(x)\, w\|_2.$$

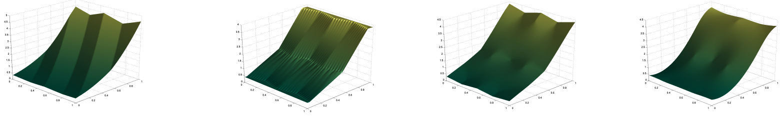
- Enforce regularity where u is almost planar, decrease where $v$ is accurate: normalize and solve

$$\min_{v'} \frac{1}{2} \int_\Omega w(x) \|v'(x) - v(x)\|_2^2 dx + \frac{\rho}{2} \int_\Omega \|Dv'(x)\|_2^2 dx.$$

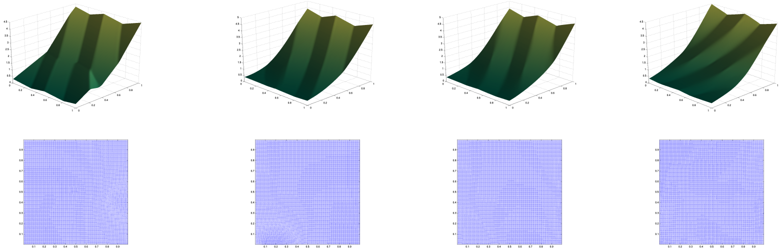
  where $w(x)$ is largest singular value of $K_\sigma * (D(Du/|Du|))(x)$.
- For unknown $u$, start with a random field $v^0$ (almost isotropic) and alternate between computing $u^k$ and $v^k$

# Adaptive choice of $v$

- Still sharp edges, ambiguities are correctly resolved



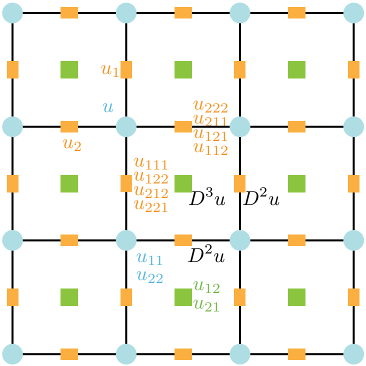
original | AMLE | $\|D^2 u\|$ | $\|D^3 u\|$



$\|D^2 u(v,\cdot)\|$, $|D^2 u(v,v)|$ | $\|D^3 u(v,\cdot,\cdot)\|$ | $\|D^3 u(v,v,\cdot)\|$ | $|D^3 u(v,v,v)|$

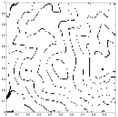
# Discretization

- Sensitive to discretization – use staggered grid:

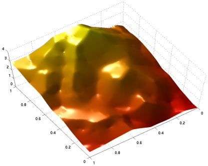


# Real-world results

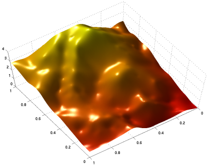
- “Bull mountain” from National Elevation Dataset [Gesch et al. ’09]



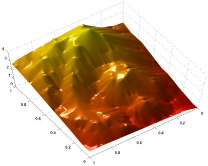
input



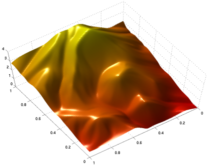
$\|D^2 u\|$



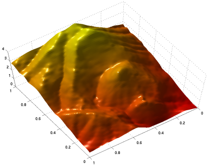
$\|D^3 u\|$



AMLE

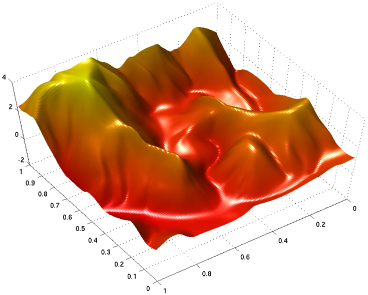


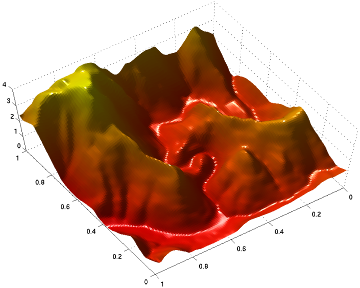
$\|D^3 u(v, \cdot, \cdot)\|$



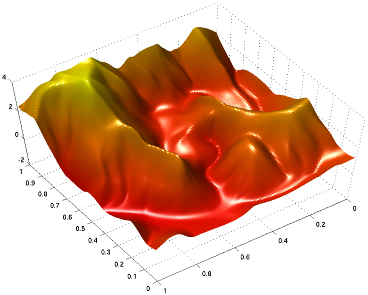
ground truth

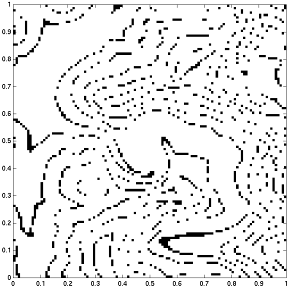
# Real-world results



# Real-world results



# Real-world performance

- $L^2$ distance

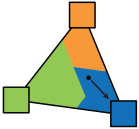
| # | quadratic | AMLE | $TV^{(3)}$ | $R_1^{(3)}$ | $R_2^{(3)}$ | $R_3^{(3)}$ | $TV^{(2)}$ | $R_1^{(2)}$ | $R_2^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 141.57 | 138.15 | 15.10 | **8.45** | 10.99 | 14.46 | 30.59 | 18.80 | 25.64 |
| 2 | 95.55 | 83.91 | 13.61 | **11.10** | 13.41 | 21.28 | 25.29 | 19.18 | 28.74 |
| 3 | 202.06 | 235.86 | 29.61 | **28.75** | 33.42 | 50.23 | 84.71 | 43.34 | 97.27 |
| 4 | 79.33 | 56.36 | 18.93 | **8.59** | 10.34 | 13.74 | 26.04 | 13.35 | 21.41 |
| 5 | 103.76 | 88.91 | 34.47 | **17.03** | 21.96 | 26.23 | 43.84 | 23.10 | 28.53 |

- $L^2$ distance of normals

| # | quadratic | AMLE | $TV^{(3)}$ | $R_1^{(3)}$ | $R_2^{(3)}$ | $R_3^{(3)}$ | $TV^{(2)}$ | $R_1^{(2)}$ | $R_2^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 10.88 | 15.33 | 3.46 | **2.12** | 2.43 | 2.95 | 4.32 | 3.35 | 5.49 |
| 2 | 21.73 | 22.94 | 6.66 | **5.77** | 7.04 | 10.85 | 9.45 | 9.07 | 13.71 |
| 3 | 29.96 | 42.91 | 10.67 | **10.49** | 12.64 | 19.12 | 15.32 | 15.02 | 21.74 |
| 4 | 13.28 | 11.39 | 5.34 | **3.31** | 3.74 | 4.76 | 6.78 | 4.44 | 6.63 |
| 5 | 9.86 | 10.36 | 5.09 | **3.15** | 3.74 | 4.65 | 5.77 | 4.41 | 5.68 |

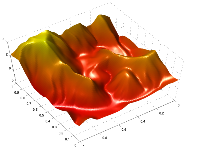
# Conclusion

- **Convex relaxation with optimality bound**
  - *Probabilistic a priori* bound
  - Approximate generalized coarea formula



$$C f(u) \geqslant \int_{\Gamma} f(\bar{u}_\gamma) d\mu(\gamma) = \mathbb{E}_\gamma f(\bar{u}_\gamma)$$

- **Non-convex variational surface interpolation**
  - Interesting problem, highlights properties of regularizers
  - Robust method to find surface and association between level lines



# Convexity and Non-Convexity in Partitioning and Interpolation Problems

Jan Lellmann

CIA/DAMTP, University of Cambridge

Joint work with: C. Schnörr, F. Lenzen (IPA/HCI, University of Heidelberg),
F. Widmann, B. Lellmann (Imperial College, London),
C. Schönlieb (DAMTP, University of Cambridge)
J.-M. Morel (ENS Cachan)
Acknowledgments: A. Bertozzi and A. Chen

IPAM, Feb 2013

# Reducing Metrics – Cyclic

- Shortest-path representation: $d(i,j) = \text{shortest\_path}(G,i,j)$
- Continuous analogue: $TV_{\mathcal{S}^1}$ for angular/orientation data

[StrekalovskiyCremers2011]

original

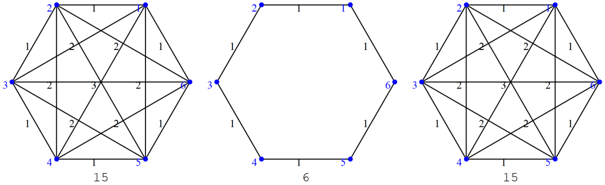
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \end{pmatrix}$$

reduced

$$\begin{pmatrix} 0 & 1 & \infty & \infty & \infty & 1 \\ 1 & 0 & 1 & \infty & \infty & \infty \\ \infty & 1 & 0 & 1 & \infty & \infty \\ \infty & \infty & 1 & 0 & 1 & \infty \\ \infty & \infty & \infty & 1 & 0 & 1 \\ 1 & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

completion

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \end{pmatrix}$$




- Can automatically reduce number of constraints: $O(n^2) \to O(n)$

[Lellmann et al. '13]

# Reducing Metrics – Multiple Components

- Separable metric (*uniform/Potts*):

$$d((i_1, i_2), (j_1, j_2)) = 1_{i_1 \neq j_1} + 1_{i_2 \neq j_2}$$

original

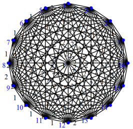
$$\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\
1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\
1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\
2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 \\
2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

reduced

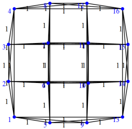
$$\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\
1 & 0 & 1 & 1 & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty \\
1 & 1 & 0 & 1 & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty \\
1 & 1 & 1 & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 \\
1 & \infty & \infty & \infty & 0 & 1 & 1 & 1 & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\
\infty & 1 & \infty & \infty & 1 & 0 & 1 & 1 & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty \\
\infty & \infty & 1 & \infty & 1 & 1 & 0 & 1 & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty \\
\infty & \infty & \infty & 1 & 1 & 1 & 1 & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 \\
1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 0 & 1 & 1 & 1 & 1 & \infty & \infty & \infty \\
\infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & 1 & 0 & 1 & 1 & \infty & 1 & \infty & \infty \\
\infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & 1 & 1 & 0 & 1 & \infty & \infty & 1 & \infty \\
\infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & 1 & 1 & 1 & 0 & \infty & \infty & \infty & 1 \\
1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 0 & 1 & 1 & 1 \\
\infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & 1 & 0 & 1 & 1 \\
\infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & 1 & 1 & 0 & 1 \\
\infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & \infty & \infty & \infty & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

completion

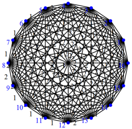
$$\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\
1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\
1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\
2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 \\
2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$



120



48



120

- Reduction: $O(n^4) \to O(n^3)$ (still $n^2$ labels)
- Also on-the-fly techniques, computationally efficient

[Chambolle/Cremers/Pock '12]