Measure-Valued Variational Image Processing

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About me

2007-2011	PhD University of Heidelberg (C. Schnörr)
2011-2015	Postdoc/Leverhulme Fellow University of Cambridge (C. Schönlieb)
2015-current	Professorship in Applied Mathematics University of Lübeck (MIC)

Measure-Valued **Variational Image Processing**

Given measurements $\boldsymbol{b}$, find image data $\boldsymbol{u}$ so that

$$b = T(u) + n$$

T structural operator, n random noise

Often the direct reconstruction is not unique, not stable, or involves hidden variables – we need prior knowledge

Variational methods

We reconstruct the unknown data $\boldsymbol{u}$ from the measurements $\boldsymbol{b}$ by minimizing the energy

$$\min_{u \in \mathcal{U}} \{D(T(u); b) + R(u)\}$$

$$TV(u) = \int_\Omega d\|Du\|$$

Advantages:

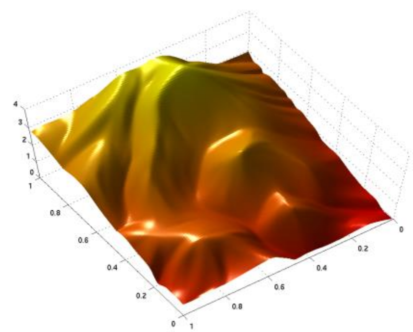
- Intuitive – we specify what the results should look like
- Often statistical motivation – maximum a posteriori estimate
- Modular, reusable components

Tikhonov '63

Measure-Valued Variational Image Processing

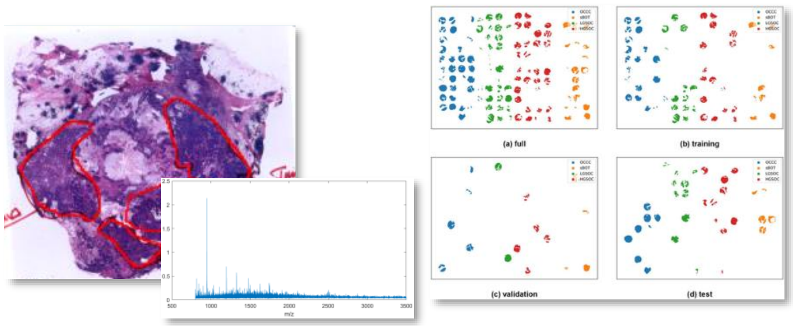
Range

- Often: $u\colon \Omega \to X \subseteq \mathbb{R}^k$



digital elevation map [1,2]
$u\colon \Omega \to \mathbb{R}$

RGB image [3]
$u\colon \Omega \to \mathbb{R}^3$



Spectral data (MALDI-MSI) [4]
$u\colon \Omega \to \mathbb{R}^K$

- Many tools: L^p, BV, (weak) derivatives, Sobolev/total variation regularization, existence, regularity, …

[1] J. Lellmann, J.-M.~Morel, C.Schönlieb '13
[2] F. Lenzen, J. Lellmann, F. Becker, C. Schnörr '14
[3] J Lellmann: Imaging Live, http://imaging.live
[4] O. Klein, F.Kanter, H. Kulbe, P. Jank, C. Denkert, G. Nebrich, W.D. Schmitt, Z. Wu, C.A. Kunze, J. Sehouli, S. Darb-Esfahani, I. Braicu, J.Lellmann, H. Thiele, E.T. Taube'18

This talk

$$\inf_{u:\Omega\to(V,\|\cdot\|_v)} D(u,I) + R(u)$$

where $(V, \|\cdot\|_v)$ is a *normed measure space.*

*needs suitable function space

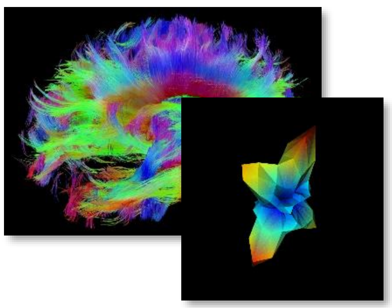
Each $u(x)$ is a measure!

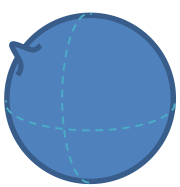
Why?

Measure spaces are linear

Embeddings

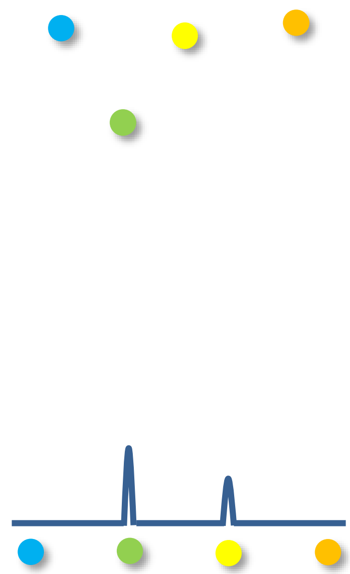
I. Naturally measures





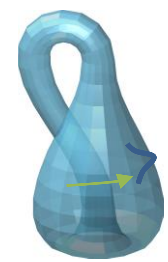
$\mathcal{P}(\mathcal{S}^2)$

II. Discrete range



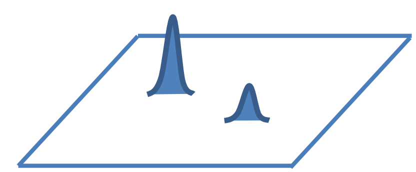
$\mathcal{P}(\{0,1,2,3\})$

III. Manifolds



$\mathcal{P}(K)$

IV. Nonconvexity



$\mathcal{P}\left((0,1)^2\right)$

Why?

Embeddings

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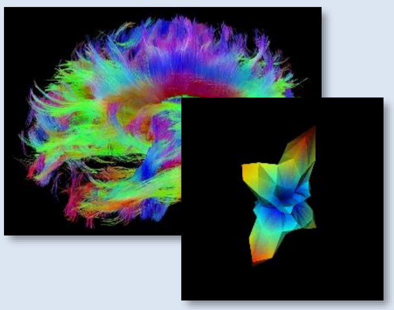

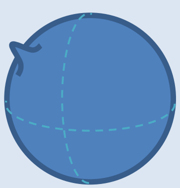

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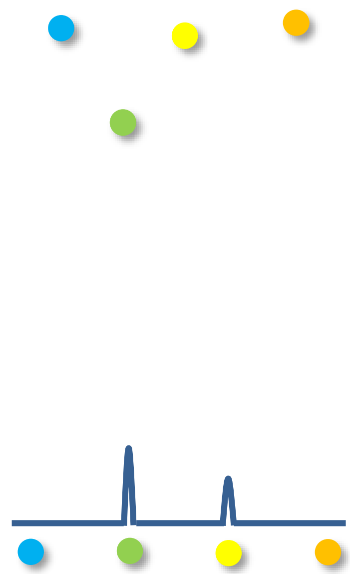

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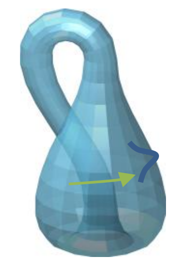

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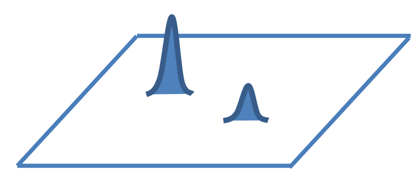
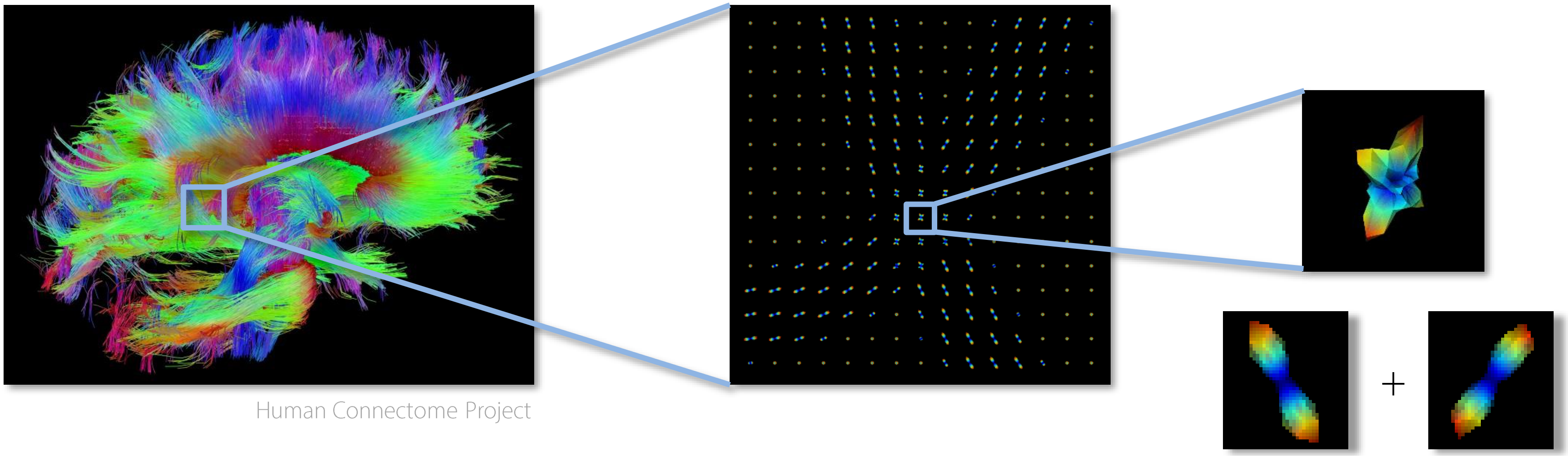


$\mathcal{P}((0,1)^2)$

Application: Diffusion-Weighted MRI



Human Connectome Project

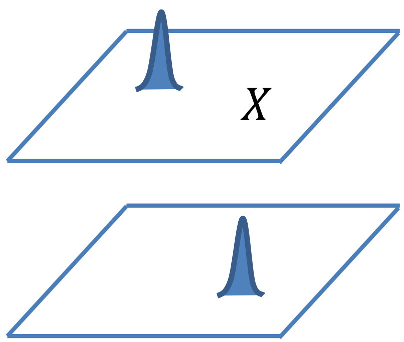
Problem: Solve

$$\min_{u:\Omega\to\mathcal{P}(\mathcal{S}^2)} D(u;b)+R(u).$$

Q-ball imaging: Tuch '04; CSA-ODF: Aganj, Lenglet, Sapiro '09; CSD: Tournier, Calamante, Connelly '07; Tax et al. '14
Alternative approach using Fisher-Rao metric: Weinmann, Demaret, Storath '14
Diffusion in Roto-Translation space: Duits, Franken '11; Portegies et al. '15

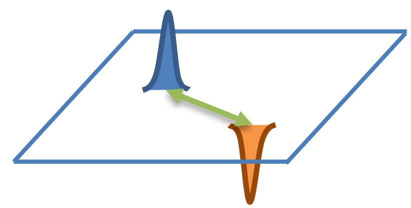
Comparing measures

- (Total) Variation metric on $\mathcal{P}(X)$:
  $d(\mu,\mu') \coloneqq \|\mu-\mu'\|_{\mathcal{M}} \coloneqq \int_X d|\mu-\mu'|$



$|\delta_a| = 1$

$|\delta_b| = 1$



$|\delta_a - \delta_b| = 2$

"L^1", ignores geometry of X!

- (1-)Wasserstein metric:

$$W_1(\mu,\mu') \coloneqq \|\mu-\mu'\|_{KR} \coloneqq$$
$$\sup\left\{\int_X p\, d(\mu-\mu') \,\middle|\, [p]_{Lip(X)} \leq 1, p(x_0) = 0\right\}$$
$$W_1(\delta_a,\delta_b) = d_X(a,b)$$

KR-norm: Kantorovich, Rubinstein '57; Melleray, Petrov, Vershik '08; Generalization: Chizat, Peyré, Schmitzer, Vialard '15; Registration: Haker et al. '04; Segmentation, shape matching: Chan, Esedoglu '09; Swoboda, Schnörr '13; Schmitzer, Schnörr '13; Image and shape retrieval: Rubner, Tomasi, Guibas '00–02; Rabin, Peyré '10; Color transfer: Rabin, Peyré '11; BRDF interpolation: Bonneel et al. '11; Shape barycenter: Rabin et al. '12; Displacement interpolation, gradient flows: McCann '97; Benamou, Brenier '00; Burger, Carillo, Wolfram '10; Düring, Mattes, Milisic '10; Papadakis, Peyré, Oudet '14; Decomposition: Burger, Franek, Schönlieb '11; Denoising: Lellmann, Lorenz, Schönlieb, Valkonen '14

Regularizer – Wasserstein-TV

- Can compare two measure-valued function respecting geometry:

$$\inf_{u:\Omega\to\mathcal{P}(X)} \int_\Omega W_1\big(u(x), b(x)\big)dx + \cdots$$

- What about regularization?

Total variation:

$$\sup_p \int_\Omega \langle u(x), -div\, p(x)\rangle dx$$

$$s.t.\, p \in C_c^1(\Omega, \mathbb{R}),$$

$$\|p(x)\|_2 \leq 1$$

[1] Vogt, Lellmann, SSVM'17; [2] Vogt, Lellmann, JMIV'18

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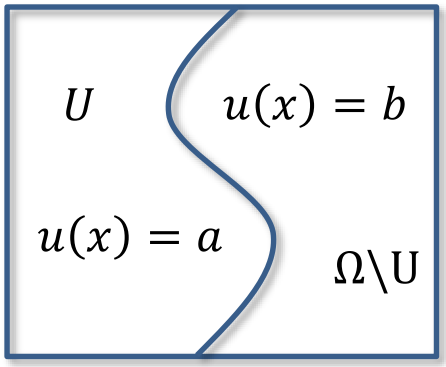

$$TV(u) = |a-b|\mathcal{H}^{d-1}(\partial U)$$

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new: **Wasserstein-TV!** [2]

$$\sup_p \int_\Omega \langle u(x), -Div\, p(x,\cdot)\rangle dx$$

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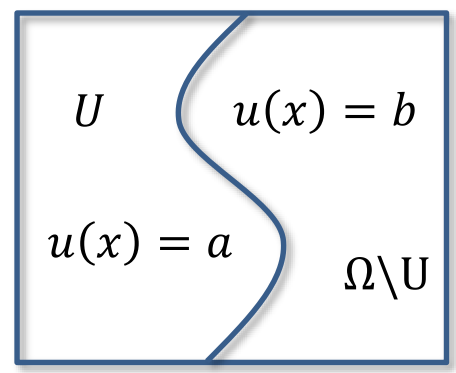
$$[p(x,\cdot)]_{Lip(X)^d} \le 1$$



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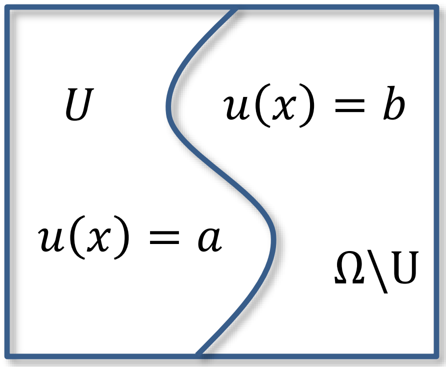

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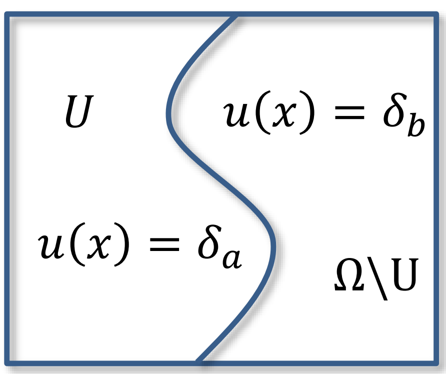

$$TV(u) = d_X(a,b)\mathcal{H}^{d-1}(\partial U)$$

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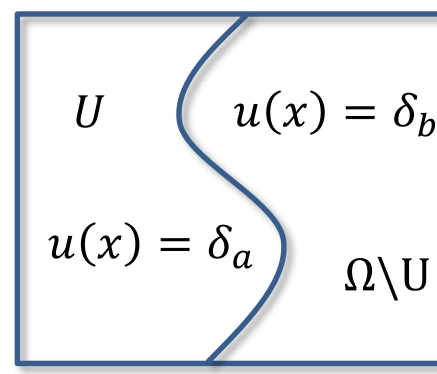
$$\sup_{a\neq b\in X^d} \frac{\|p(x,a) - p(x,b)\|_2}{d_x(a,b)}$$



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Norms and measure spaces on X

Total variation $\|\mu\|_{\mathcal{M}}$

- forgets metric on X
- $(\mathcal{M}(X), \|\mu\|_{\mathcal{M}})$ is a Banach space
- $(\mathcal{M}(X), \|\mu\|_{\mathcal{M}}) = C(X)^*$ with $\|\cdot\|_\infty$

Kantorovich-Rubinstein $\|\mu\|_{KR}$

- recovers metric on X [1]
- $(KR(X), \|\cdot\|_{KR})$ is a Banach space with $KR(X) := (\widehat{\mathcal{M}_0(X), \|}\cdot\|_{KR})$
- $\mu \mapsto \mu - \delta_{x_0}$ embeds $\mathcal{P}(X)$ into $KR(X)$
- Metrizes weak* topology on $\mathcal{M}(X) = C(X)^*$
- $(KR(X), \|\cdot\|_{KR})^* = (Lip_0(X), \|\cdot\|_{Lip})$

[1] Kantorovich, Rubinstein'57; textbooks: Hewitt, Stromberg'65, Weaver'99, Villani'09

Banach space-valued TV

- $\Omega \subseteq \mathbb{R}^d$ open & bounded, V Banach space
- $u\colon \Omega \to V$ weakly measurable: $x \mapsto \langle p, u(x)\rangle$ meas. $\forall p \in V^*$

Banach space-valued TV:

$$\sup_p \int_\Omega \langle u(x), -\operatorname{div} p(x)\rangle dx$$
$$s.t.\, p \in C_c^1\left(\Omega, (V^*)^d\right),$$
$$\|p(x)\|_{(V^*)^d} \leq 1$$

- Lemma [1]: Integrals are well-defined
- Special case Wasserstein-TV:

$$V = KR(X), \qquad V^* = (Lip_0(X), \|\cdot\|_{Lip})$$

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- $(\mathcal{M}(X), \|\mu\|_{\mathcal{M}})$ is a Banach space
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- Not separable unless X discrete (⇒ not compact)

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- **Compact!** → existence

[1] Kantorovich, Rubinstein'57; textbooks: Hewitt, Stromberg'65, Weavcer'99, Villani'09

Application: DW-MRI – Results

Theorem [1]

Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let (X, d) be a compact metric space. For any $b \in L_w^\infty(\Omega, \mathcal{P}(X))$ and $\lambda \geq 0$, the variational problem

$$\inf_u \int_\Omega W_1\big(u(x), b(x)\big)dx + \lambda TV_{W_1}(u)$$

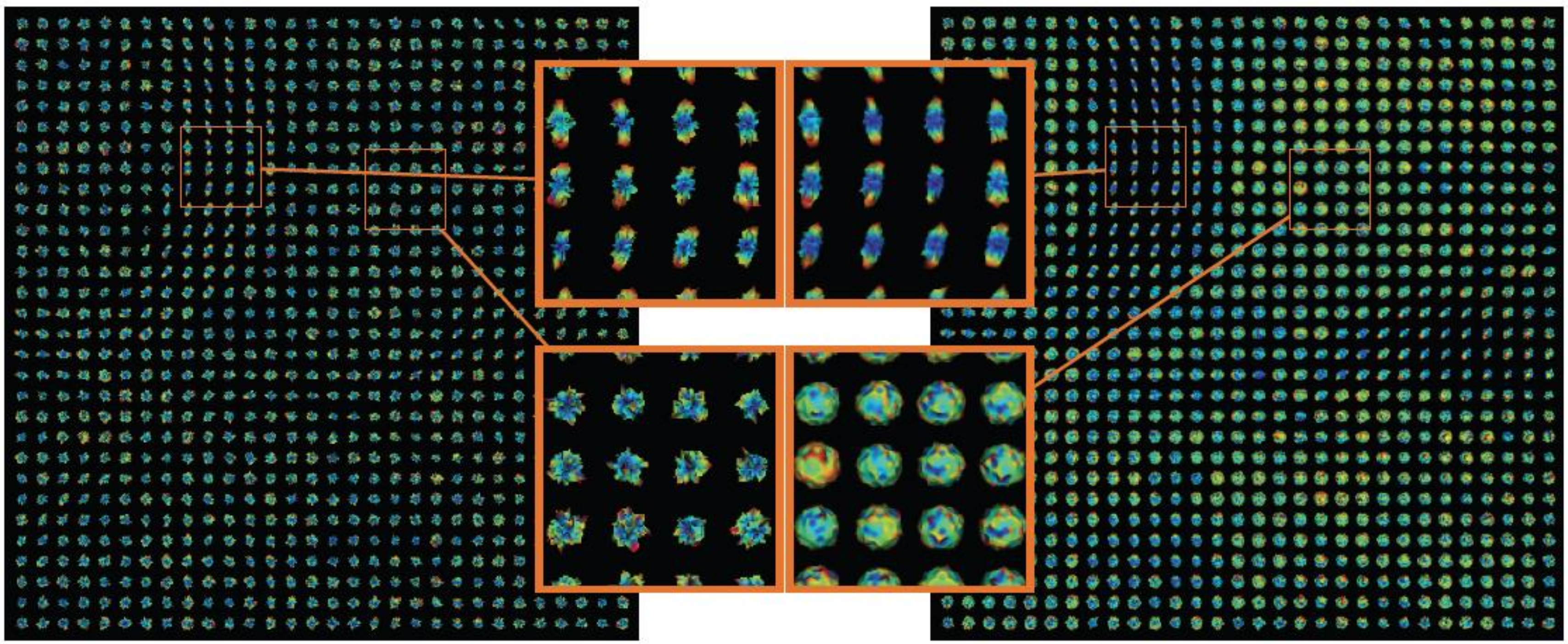
admits a (generally non-unique) solution in $L_w^\infty(\Omega, \mathcal{P}(X))$.

Roadmap:

- Need weak* measurability
- Mass-bounded minimizing sequence
- Banach-Alaoglu gives weakly* converging subsequence
- Prove weak* lower-semicontinuity

[1] Vogt, Lellmann, JMIV'18; Young measures: [2] Ball'89

Application: DW-MRI – Results



Vogt, Lellmann, JMIV'18

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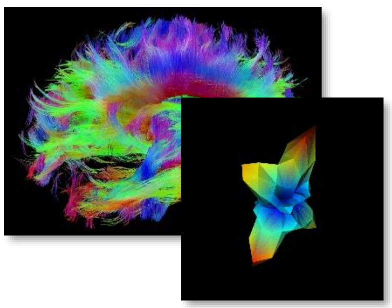

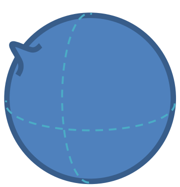

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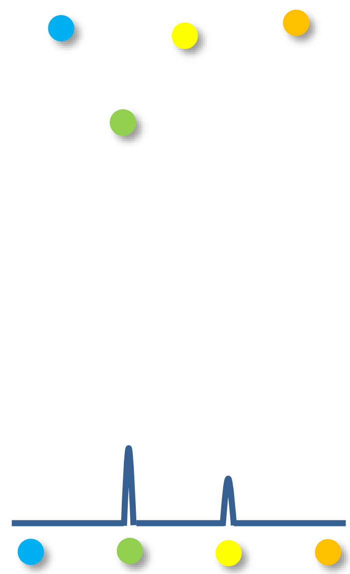

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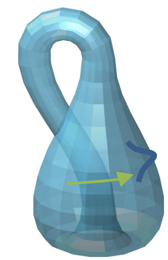

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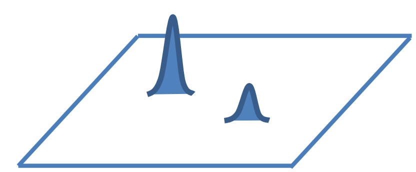

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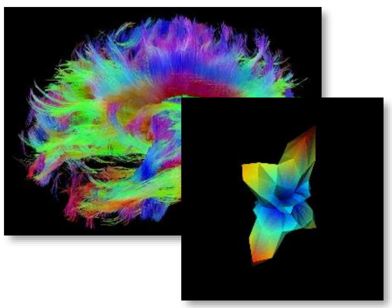

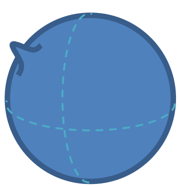

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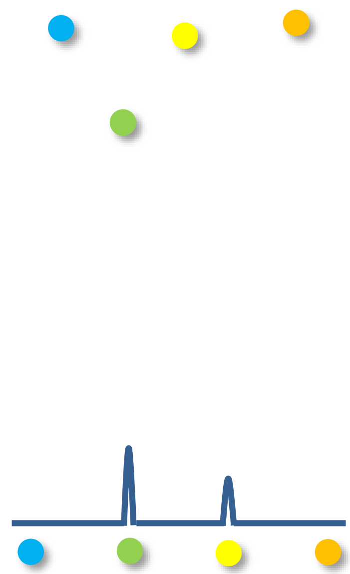

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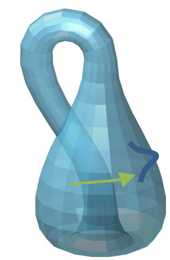

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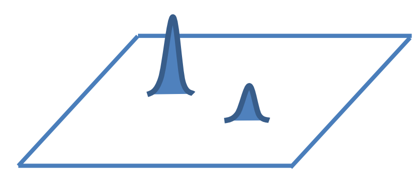


$\mathcal{P}((0,1)^2)$

Embeddings

- Idea: range X is not "nice":

$$\inf_{\mathbf{u}:\Omega\to X} \int_\Omega \rho\big(x, u(x)\big)dx + \lambda\, TV_X(u)$$

embed $X \hookrightarrow \mathcal{P}(X)$

$$\inf_{\mathbf{u}:\Omega\to\mathcal{P}(X)} \int_\Omega \int_X \rho(x,t)\,d\boldsymbol{u}_x(t)\,dt\,dx + \lambda\, TV_{W_1}(u)$$

$\delta_{u(x)}$

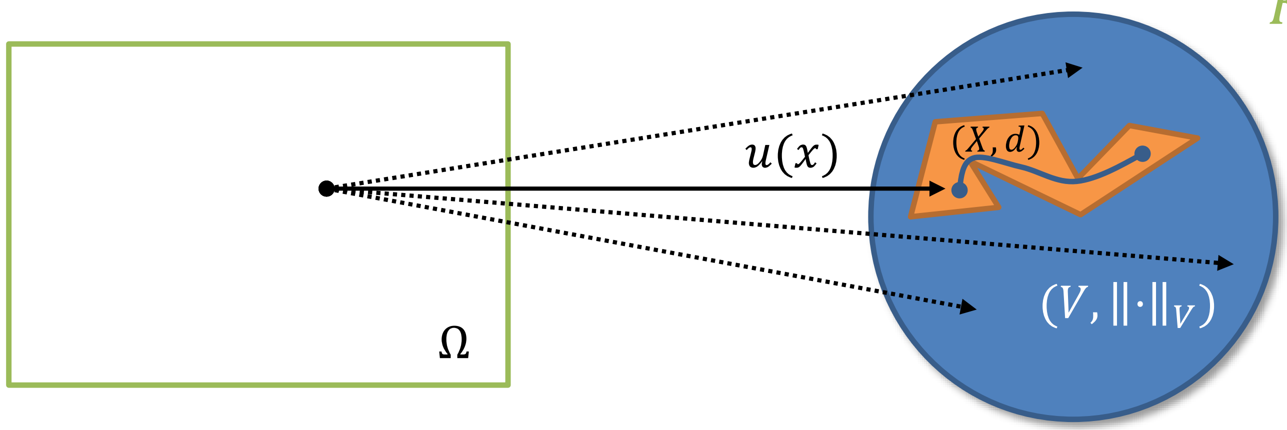
$\rho(x, u(x))$

Geometric → measure-valued

Linear spaces are much easier. Linearize!

Idea: Isometrically (!) embed $X \hookrightarrow V$
into *linear, normed* space $(\mathcal{M}(X), \|\cdot\|_V)$

$KR, \|\cdot\|_{KR}$



Need to "lift" models/functionals (can be good if nonconvex!)

$$\min_{u:\Omega\to X} f(u) \quad \Rightarrow \quad \min_{u':\Omega\to V=\mathcal{P}(X)} f'(u')$$

Tricky: Discretization? Convexity? Artificial minimizers?

Young measures: Young '37; Currents: Schwartz '51; de Rahm '55; Federer '69; Paired calibrations: Brakke '91; Alberti, Bouchitté, Dal Maso '01; DW-MRI: Vogt, Lellmann'18

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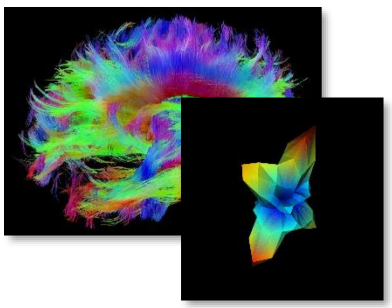
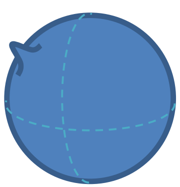
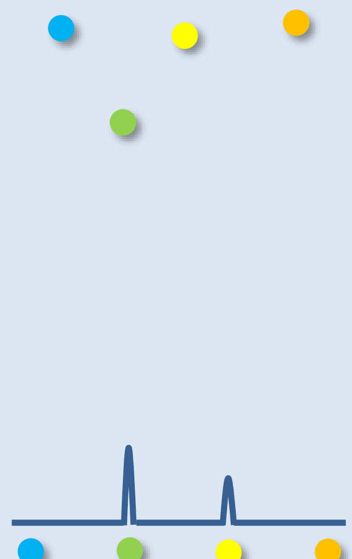




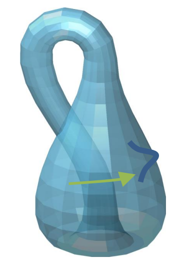
$\mathcal{P}(\mathcal{S}^2)$

II. Discrete range



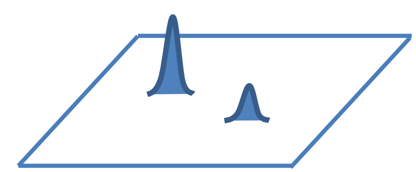
$\mathcal{P}(\{0,1,2,3\})$

III. Manifolds



$\mathcal{P}(K)$

IV. Nonconvexity



$\mathcal{P}((0,1)^2)$

Example: Image Segmentation

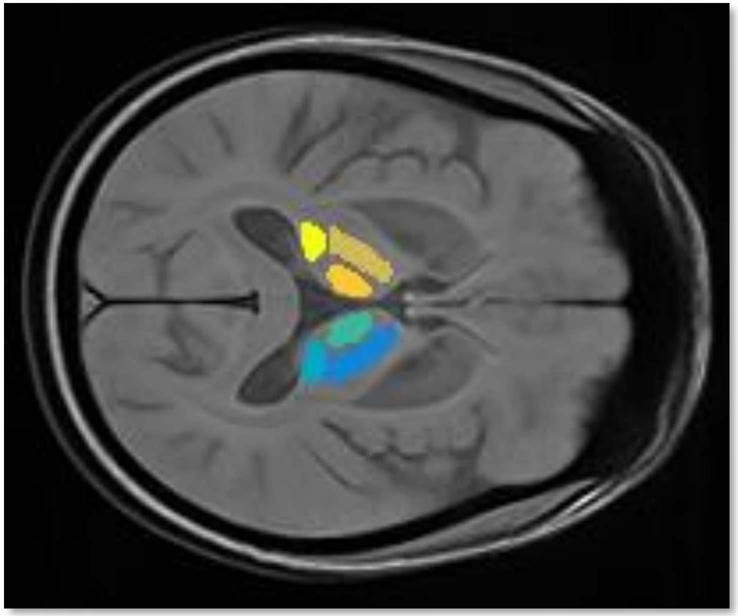
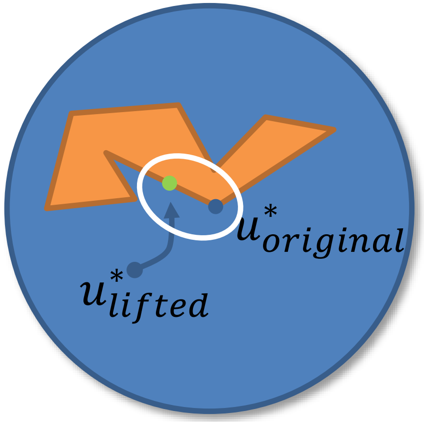


image segmentation [1]
$u: \Omega \to (\{1,2,3,4\}, d_X) =: (\boldsymbol{X}, \boldsymbol{d_X})$

derivatives?
optimization?

- $\mathcal{P}(X)$ easy to discretize (simplex, LP)
- …but $\boldsymbol{L^2}$ Lipschitz constraints
- Can prove bound [2]:

$$\mathbb{E}_\gamma f\left(round_\gamma(u^*_{lifted})\right) \leq 2C_d f(u^*_{orig})$$



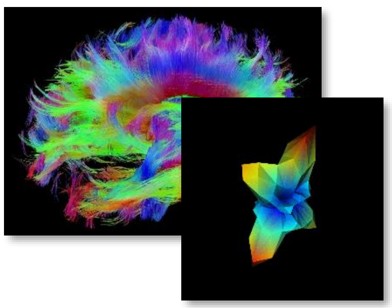
[1] Corona, Lellmann, Nestor, Schönlieb, Acosta-Cabronero

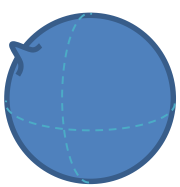
Model - finite-dimensional: Potts '52; Boykov et al. '98, '01; Kleinberg, Tardos '01; function space: Zach et al. '08; Lellmann, Becker, Schnörr '09; Chambolle, Cremers, Pock '11; Continuous min-cut/max-flow: Yuan, Bae, Tai, Boykov'10-11; Boyd, Bae, Tai, Bertozzi'18; Bounds - two-class: Strang '83; Chan, Esedoglu, Nikolova '06; Zach et al. 09, Olsson et al. 09; Finite-dimensional: Dahlhaus et al. '94, Kleinberg, Tardos '99, Boykov et al. '01, Komodakis Tziritas '07; [2] Multi-class: Lellmann, Lenzen, Schnörr, J. Math. Imag. Vis. 2013

Why?

Embeddings

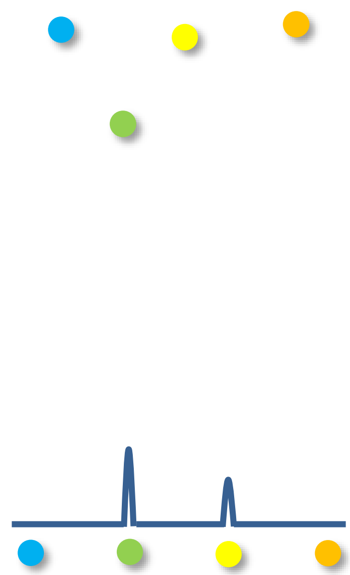
I. Naturally measures





$\mathcal{P}(\mathcal{S}^2)$

II. Discrete range

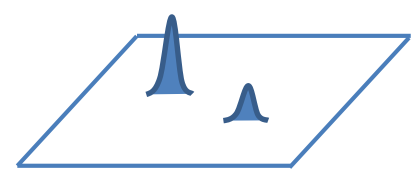


$\mathcal{P}(\{0,1,2,3\})$

III. Manifolds

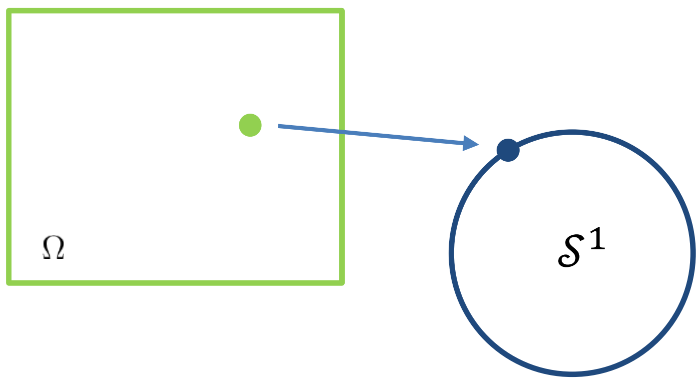
$\mathcal{P}(K)$

IV. Nonconvexity



$\mathcal{P}((0,1)^2)$

Example: Manifold-valued problems



manifold-valued models

$$\int_\Omega d_M(u(x), b(x))^p \, dx + \lambda TV_M(u)$$

range is non-convex

- Can only efficiently discretize $\mathcal{P}(X)$ for low-dimensional manifolds
- But: Lipschitz constraints

$$\|p(x, z_1) - p(x, z_2)\|_2 \leq d_{\mathcal{M}}(z_1, z_2) \ \ \forall z_1, z_2 \in M$$

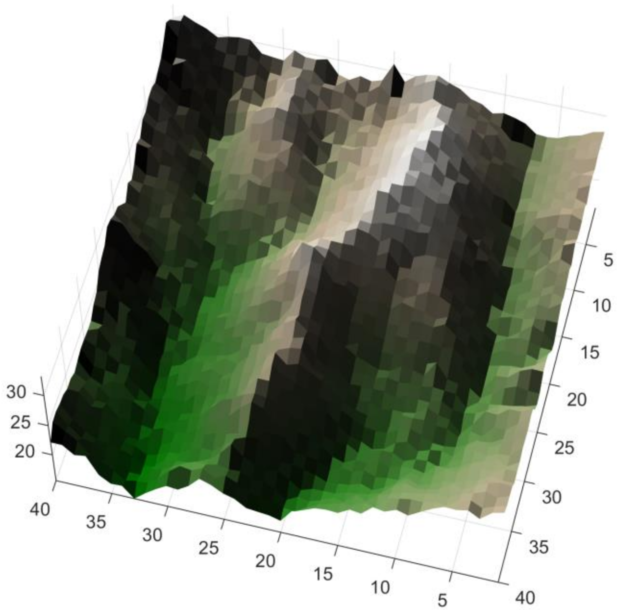
can be enforced locally [1]:

$$|D_z p(x, \cdot)|_\sigma \leq 1 \ \ \forall z \in M$$

[1] Lellmann, Strekalovskiy, Kötter, Cremers '13

Manifolds – local optimization: see Absil'07; TV: Weinmann, Demaret, Storath'14; Bacák, Bergmann, Steidl, Weinmann'16; Bergmann, Fitschen, Persch, Steidl'18

Example: Manifold-valued problems



$\mathcal{S}^2$

Lellmann, Strekalovskiy, Kötter, Cremers '13; Sketch-based 3D: see also Wu, Rahman, Tai'16

Manifold TV: Weinmann, Demaret, Storath'14:; Bacák, Bergmann, Steidl, Weinmann'16; Bergmann, Fitschen, Persch, Steidl'18

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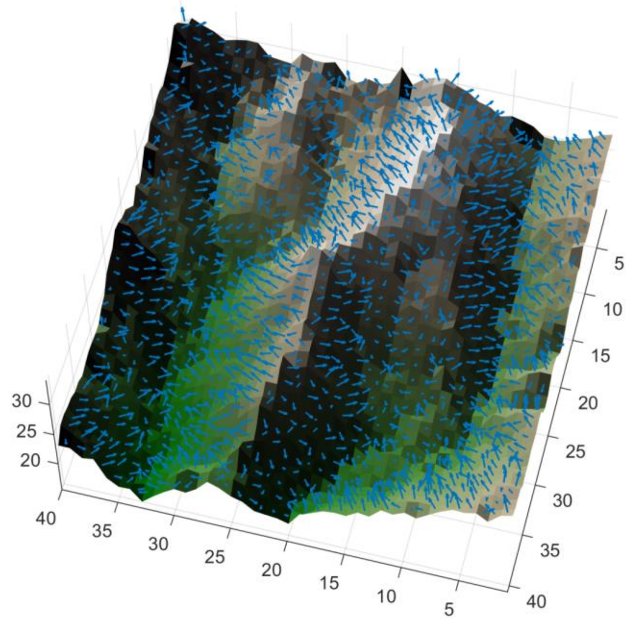

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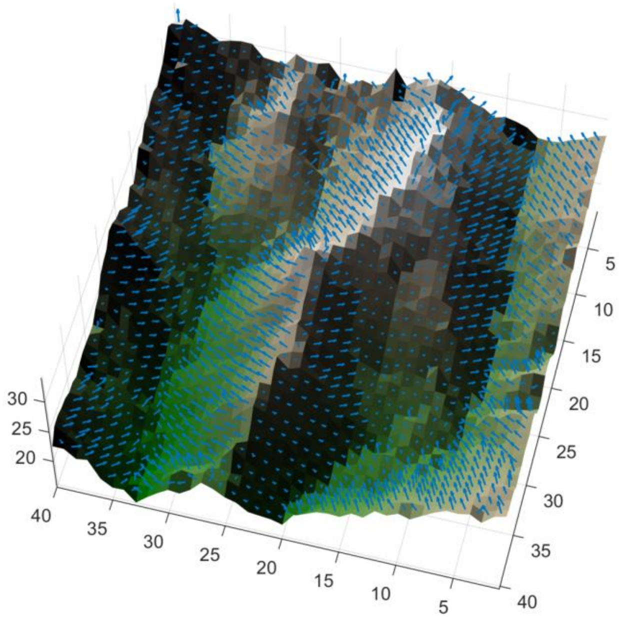

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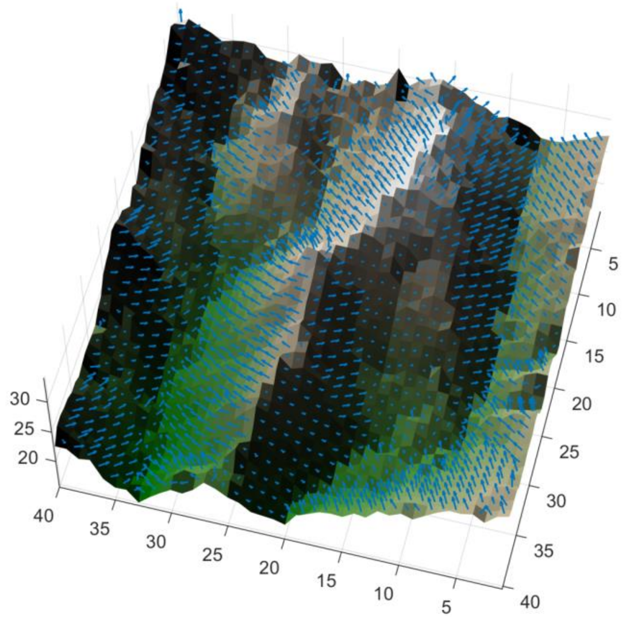
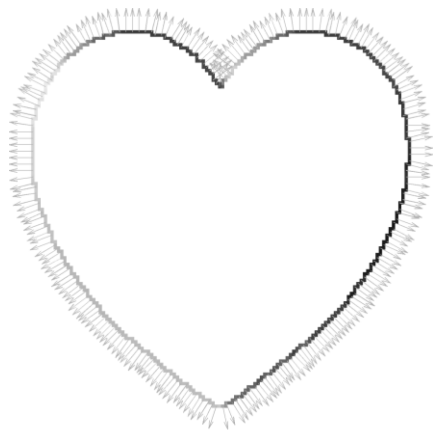


$\mathcal{S}^2$



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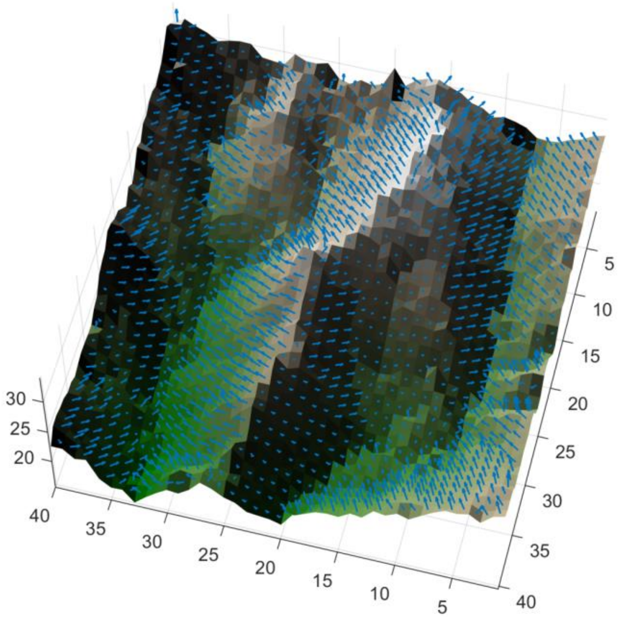

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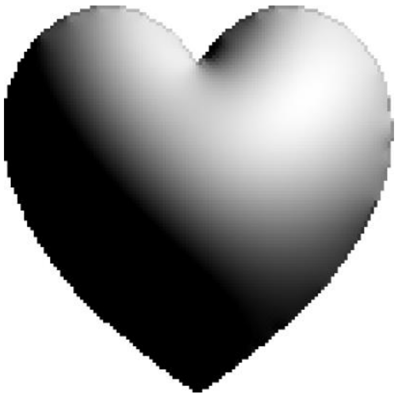

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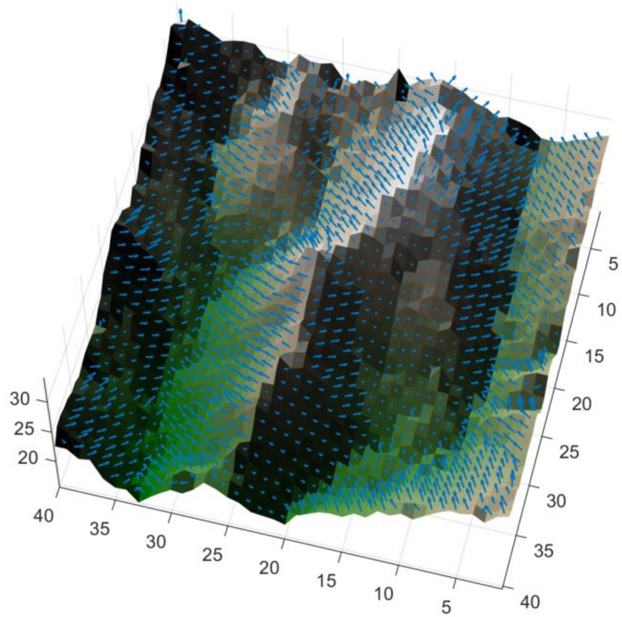

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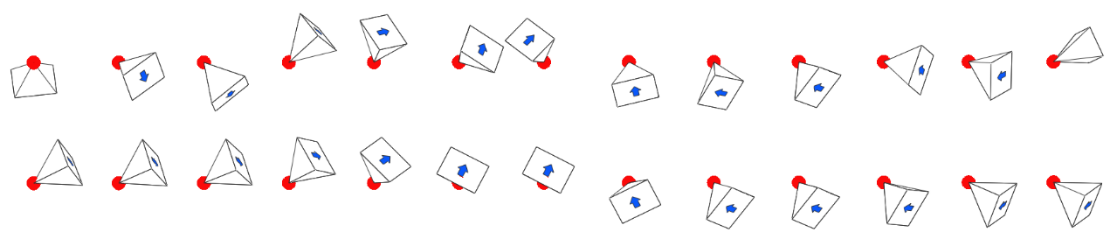

$SO(3)$

Lellmann, Strekalovskiy, Kötter, Cremers '13; Sketch-based 3D: see also Wu, Rahman, Tai'16

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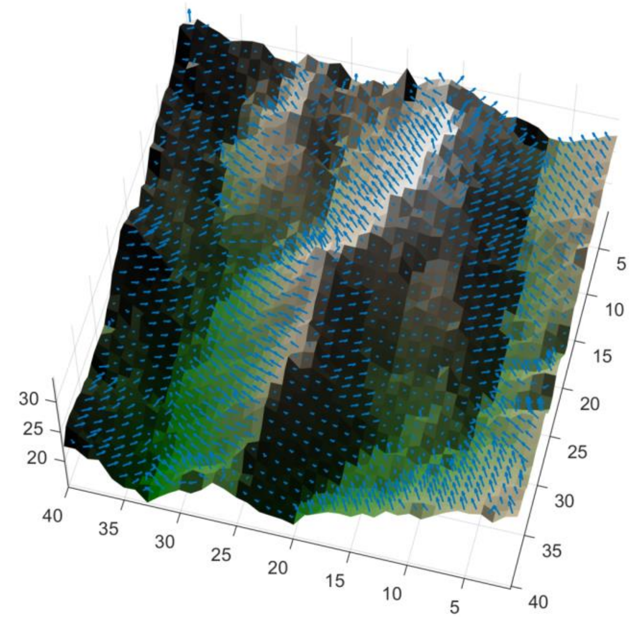
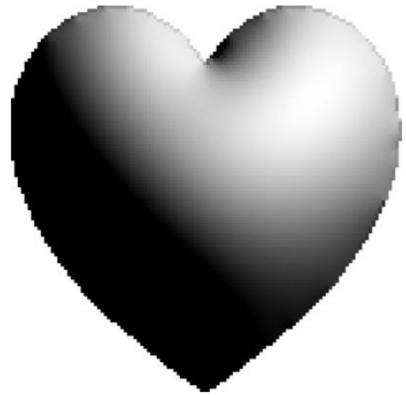


$\mathcal{S}^2$



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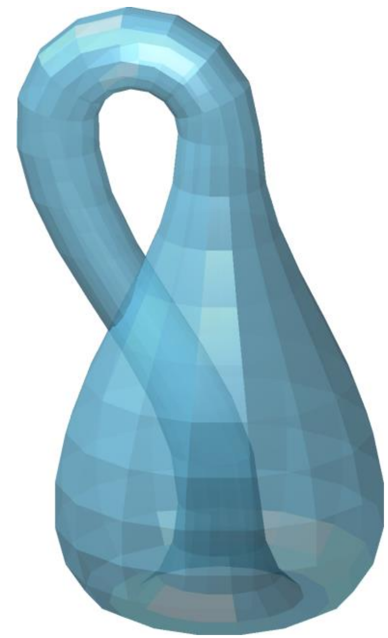
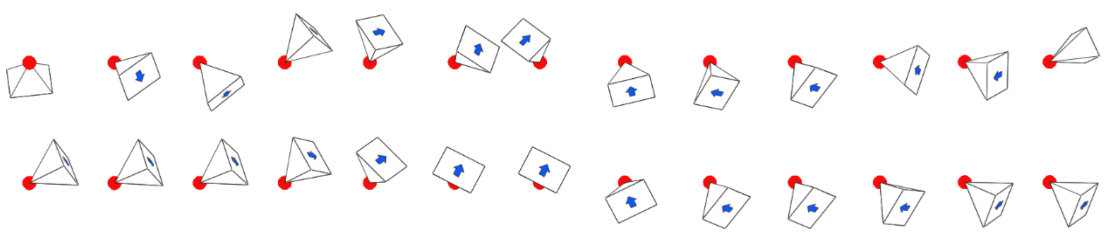


K



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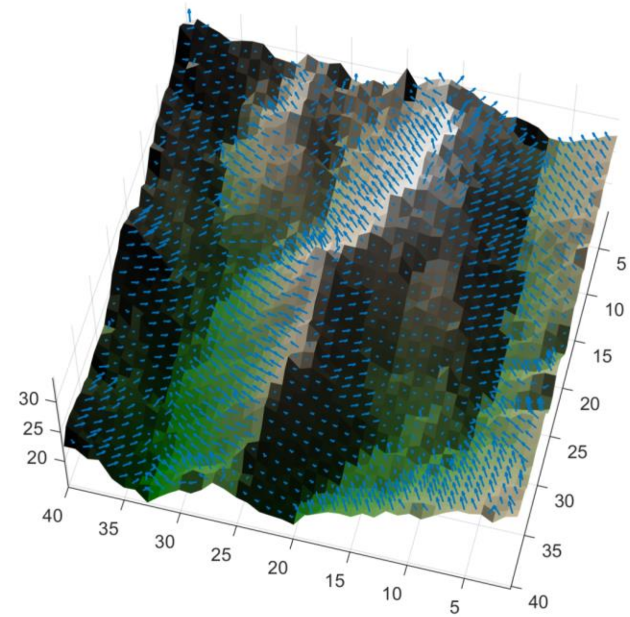

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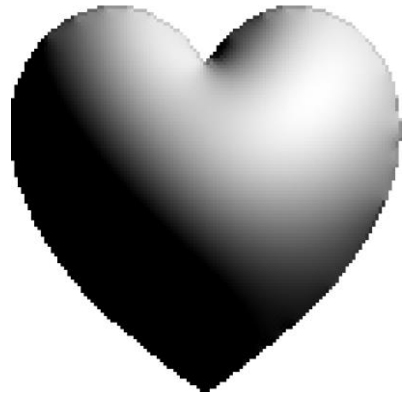

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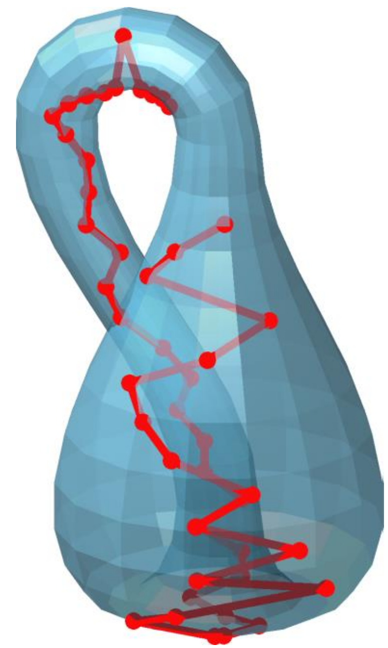

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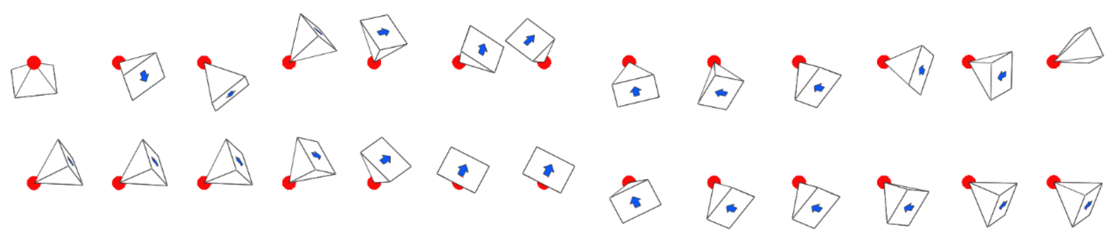

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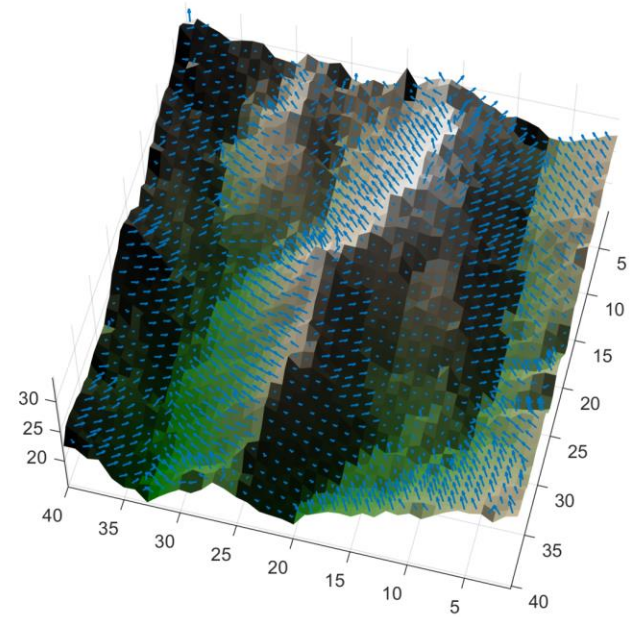

$\mathcal{S}^2$

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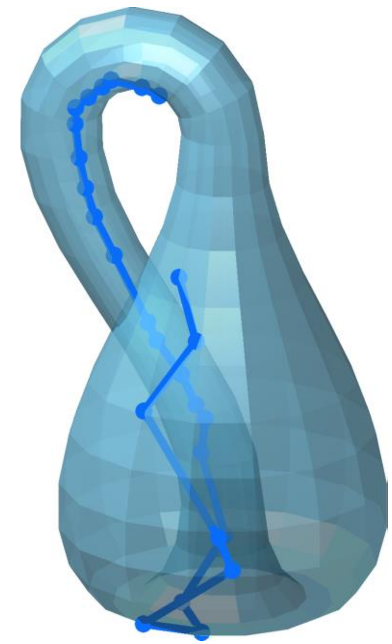
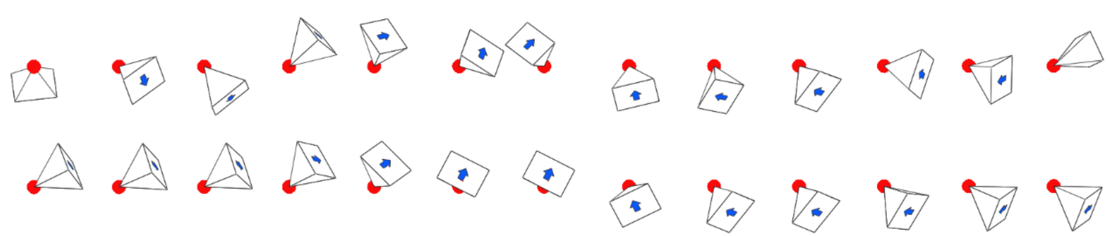


K



$SO(3)$

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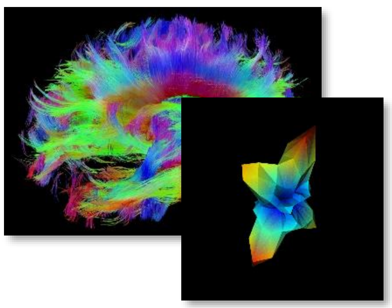
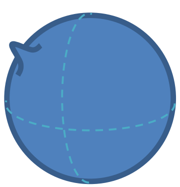
Why?

Embeddings

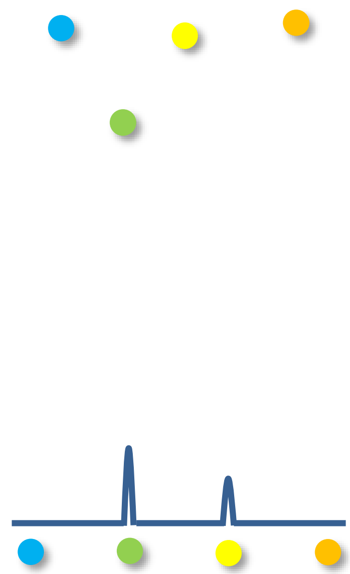
I. Naturally measures





$\mathcal{P}(\mathcal{S}^2)$

II. Discrete range

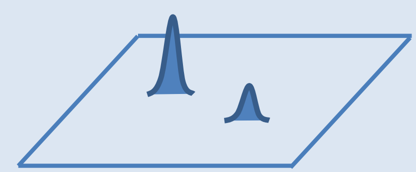


$\mathcal{P}(\{0,1,2,3\})$

III. Manifolds

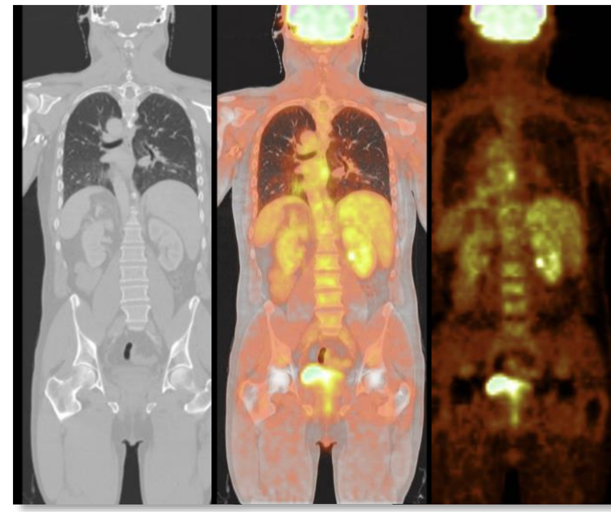
$\mathcal{P}(K)$

IV. Nonconvexity

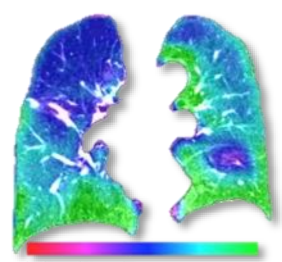


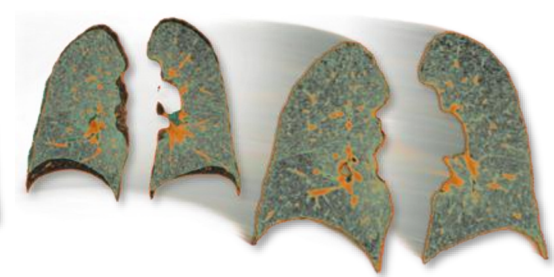
$\mathcal{P}\left((0,1)^2\right)$

MIC @ Lübeck



MIC
Institute of Mathematics and Image Computing

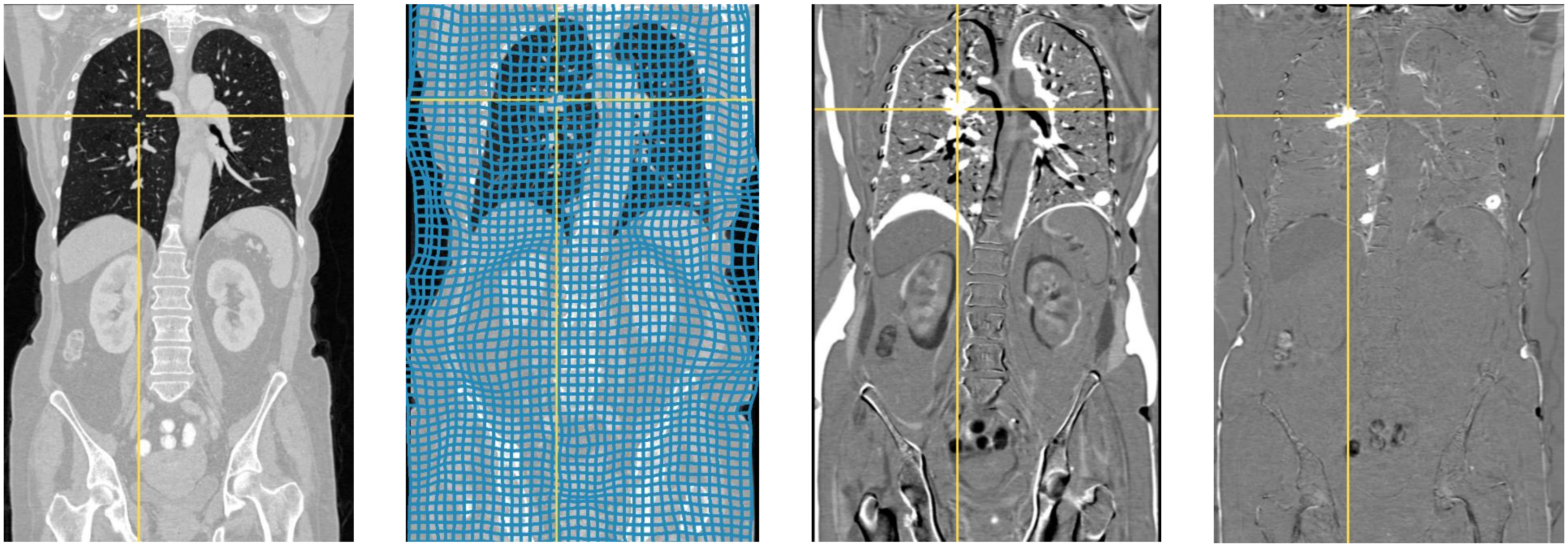




Fraunhofer
MEVIS

Fraunhofer MEVIS

Motivation – Motion Estimation



CT before | CT after | without registration | with registration [1]

$$\min_{u:\Omega\to\mathbb{R}^2} \int_\Omega D\big(I_\sigma^1(x), I_\sigma^2(x + u(x))\big)dx + \lambda\, R(u)$$

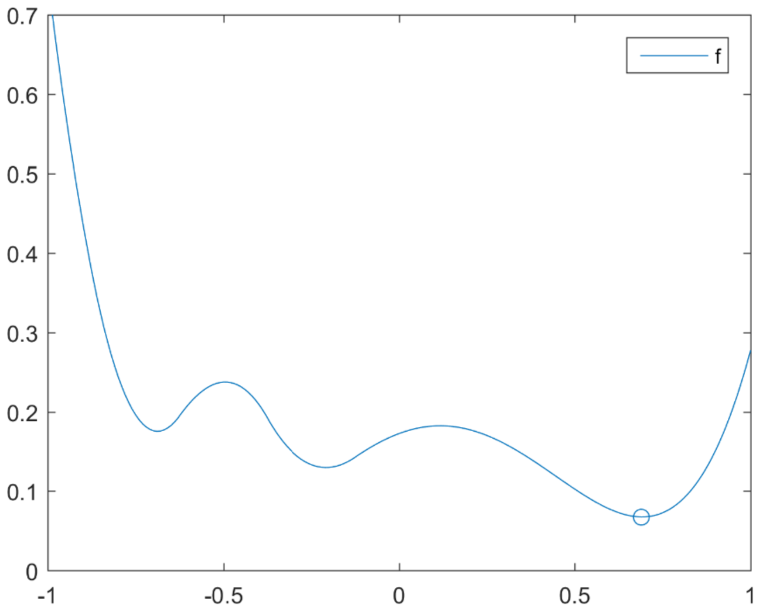
u vector field (optical flow, image registration) or scalar (disparity, stereo)

(Why) embed $\mathbb{R}^2 \hookrightarrow \mathcal{P}(\mathbb{R}^2)$?

Non-convex models

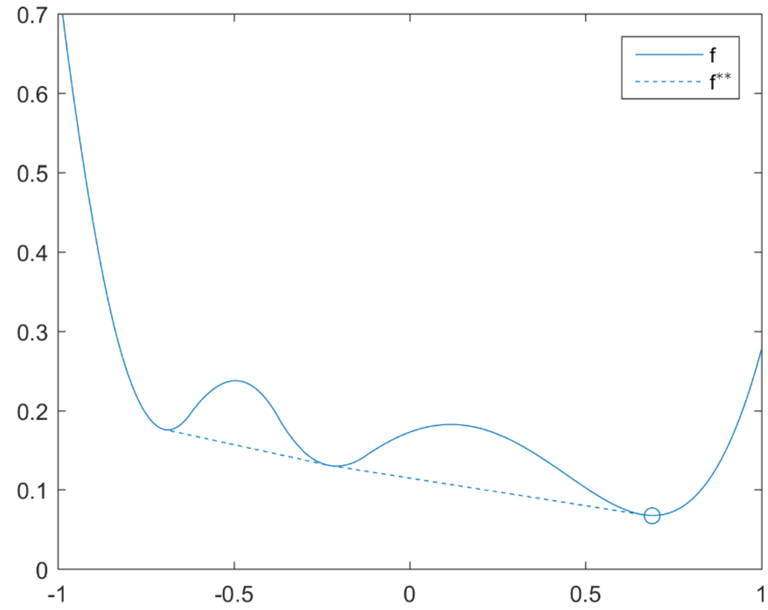
Problem:

$$\inf_{u:\Omega\to\mathbb{R}^k} \int_\Omega \rho(x, u(x))\, dx + \cdots$$



"Perfect" convexification:

$$\inf_{u:\Omega\to\mathbb{R}^k} \left(\int_\Omega \rho(x, u(x)) dx \right)^{**} (u)$$



Generally too hard!

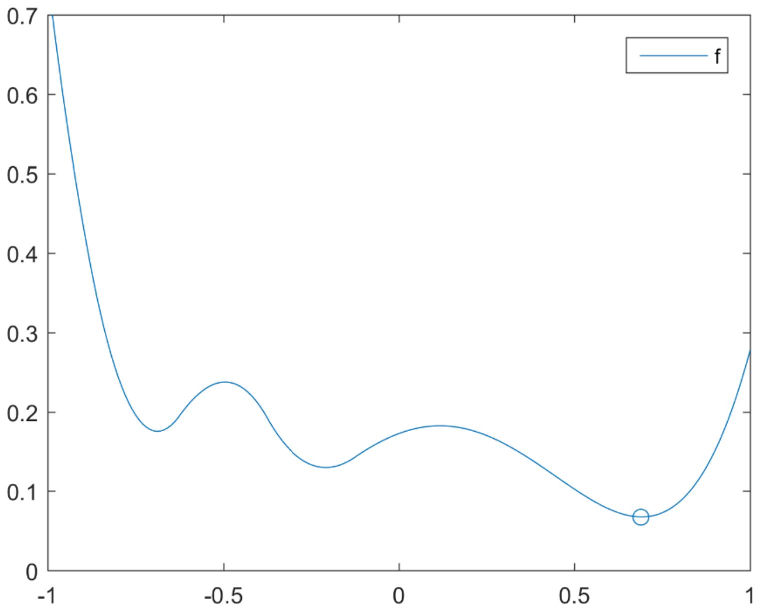
Non-convex models

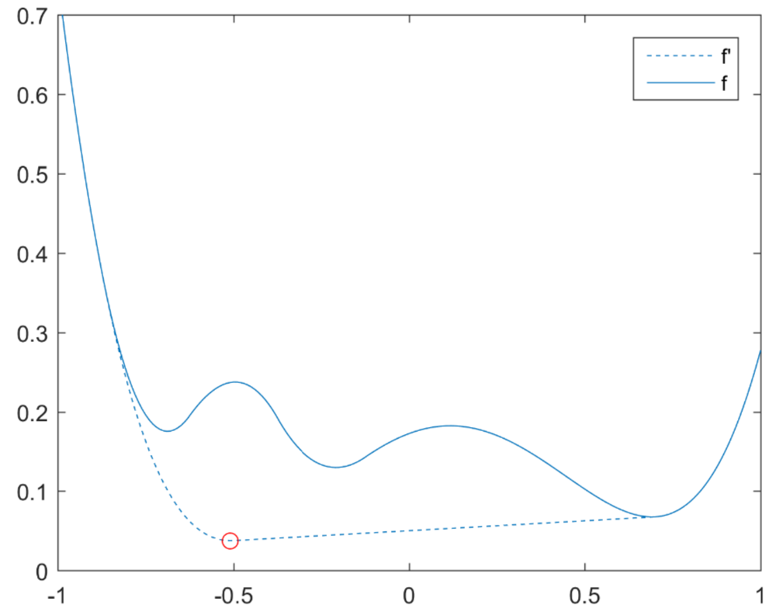
Problem:

$$\inf_{u:\Omega\to\mathbb{R}^k} \int_\Omega \rho(x, u(x))\, dx + \cdots$$

Local convexification:

$$\inf_{u:\Omega\to\mathbb{R}^k} \int_\Omega \rho^{**}(x, u(x))\, dx$$



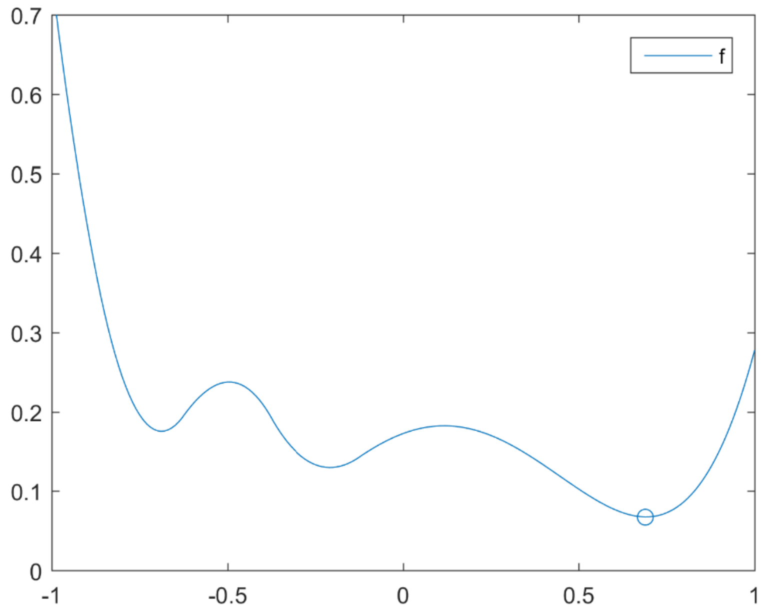


Inexact!

Non-convex models

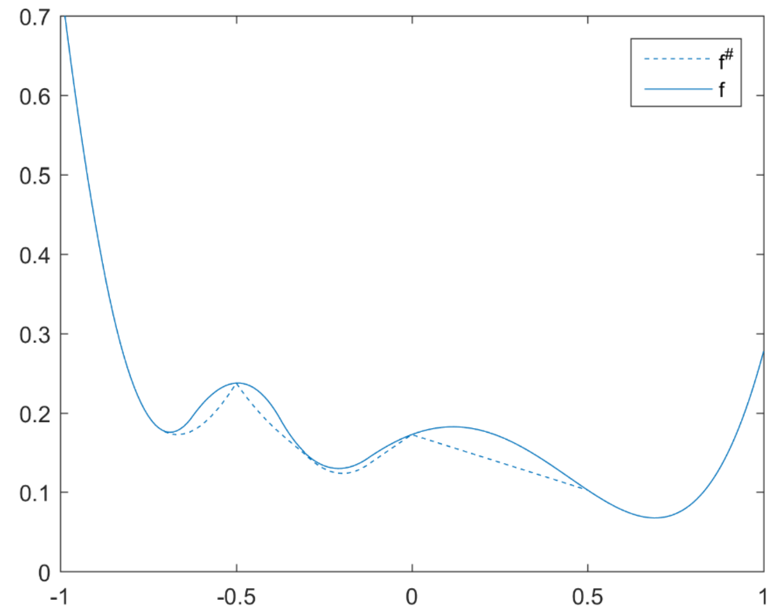
Problem:

$$\inf_{u:\Omega\to\mathbb{R}^k} \int_\Omega \rho(x, u(x))\,dx + \cdots$$



Embedding + local:

$$\inf_{\boldsymbol{u}:\Omega\to\mathcal{P}(\mathbb{R}^k)} \int_\Omega \boldsymbol{\rho}^{**}(x, \boldsymbol{u}(x))\,dx$$

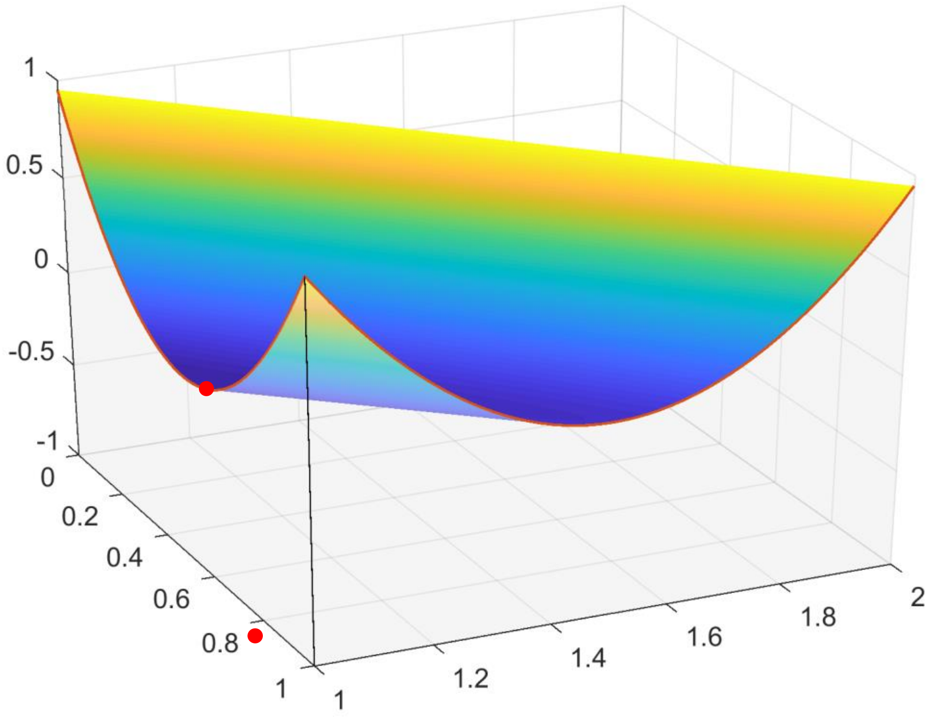


much better!

Lifting: Brakke '91; Alberti, Bouchitté, Dal Maso '01; Pock, Cremers, Bischof, Chambolle'10

Sublabel-accurate: Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16; related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

Why?



Relaxing the regularizer – TV

Proposition 4. *The convex envelope of (15) is*

$$\Phi^{**}(g) = \sup_{q \in \mathcal{K}} \langle q, g \rangle, \qquad (17)$$

where $\mathcal{K} \subset \mathbb{R}^{k \times d}$ *is given as:*

$$\mathcal{K} = \Big\{ q \in \mathbb{R}^{k \times d} \;\Big|\; \left| q^\top (1_i^\alpha - 1_j^\beta) \right|_2 \leq \left| \gamma_i^\alpha - \gamma_j^\beta \right|, \quad \forall\, 1 \leq i \leq j \leq k,\ \forall \alpha, \beta \in [0,1] \Big\}. \qquad (18)$$

Need infinitely many "Lipschitz" constraints for (locally) best embedding…

…but we can do it efficiently! [1]

Proposition 5. *In case the labels are ordered, i.e.,* $\gamma_1 < \gamma_2 < \ldots < \gamma_L$, *then the constraint set* $\mathcal{K}$ *from Eq. (36) is equal to*

$$\mathcal{K} = \{ q \in \mathbb{R}^{k \times d} \mid |q_i|_2 \leq \gamma_{i+1} - \gamma_i,\ \forall i \}. \qquad (19)$$

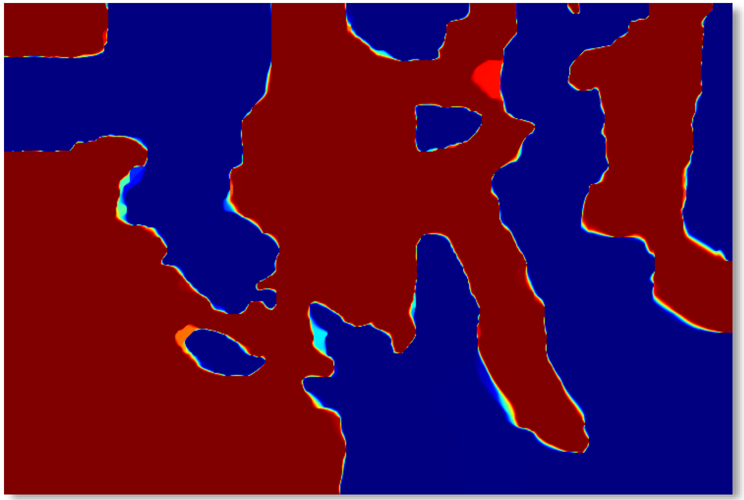
[1] Möllenhoff, Laude, Möller, **Lellmann**, Cremers, CVPR'16

Results – disparity estimation

Results – disparity estimation

linear lifting

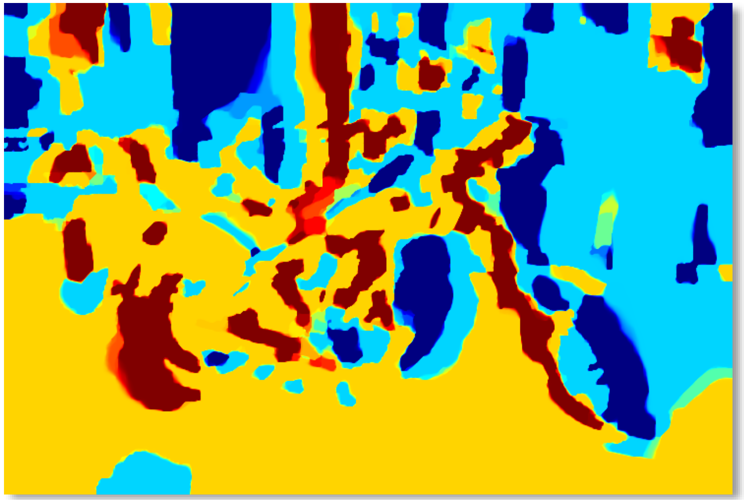
$L = 2$



Results – disparity estimation

linear lifting

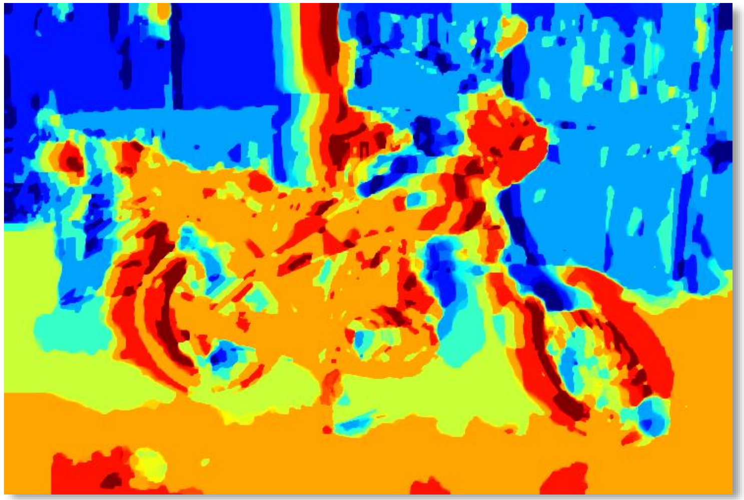
$L = 4$



Results – disparity estimation

linear lifting

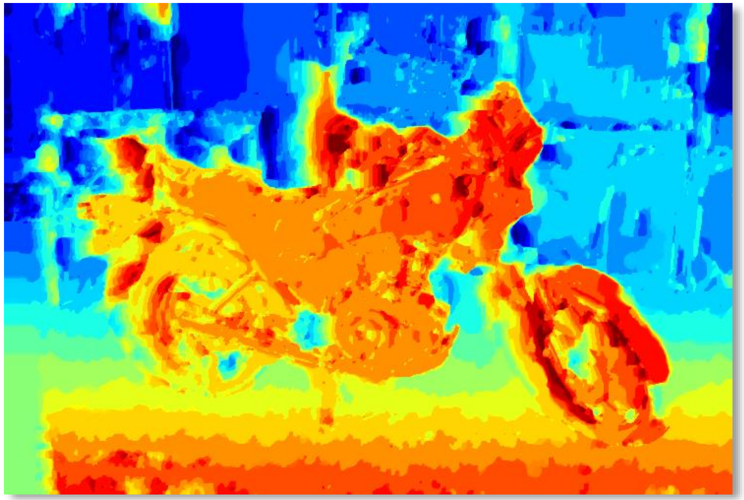
$L = 8$



Results – disparity estimation

linear lifting

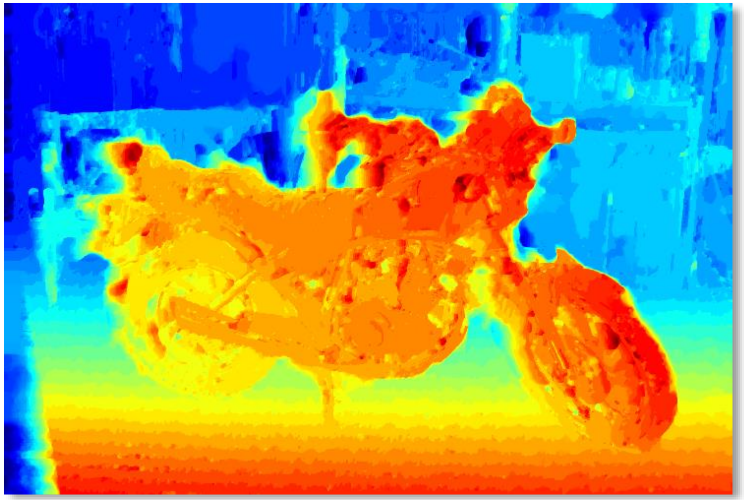
$L = 16$



Results – disparity estimation

linear lifting

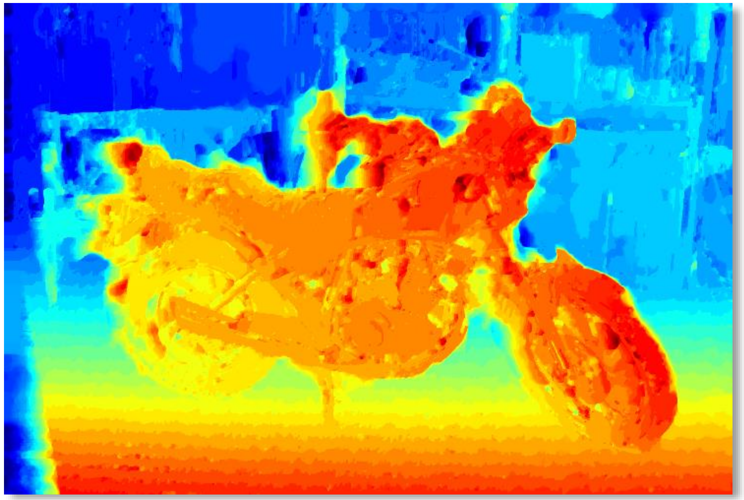
$L = 32$



Results – disparity estimation

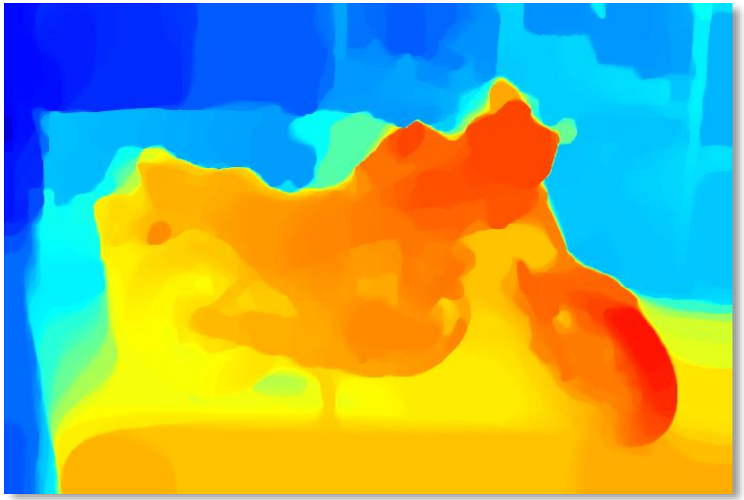
linear lifting

$L = 32$



precise lifting

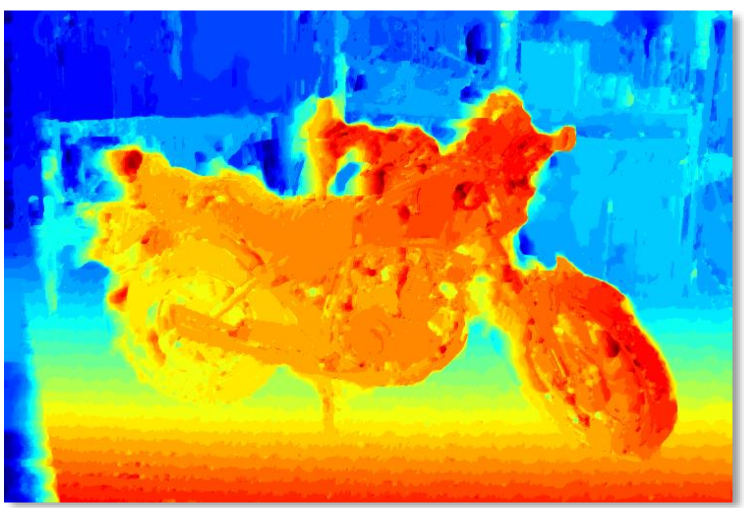
$L = 2$



Results – disparity estimation

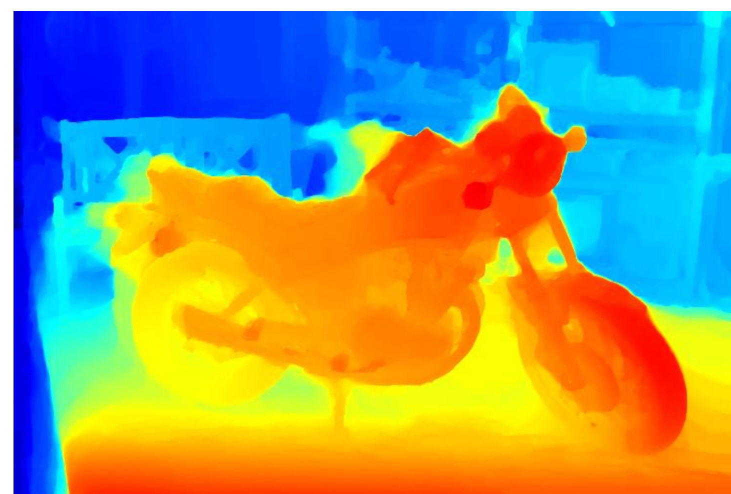
linear lifting

$L = 32$



precise lifting

$L = 4$

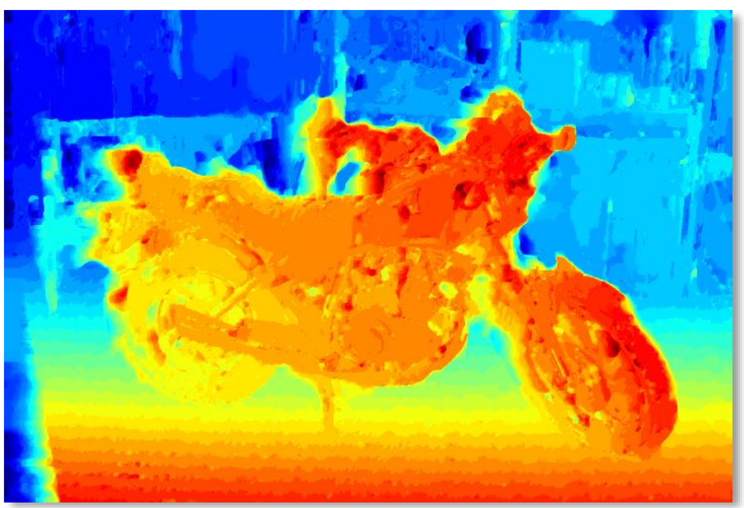


Möllenhoff, Laude, Möller, **Lellmann**, Cremers, CVPR'16

Results – disparity estimation

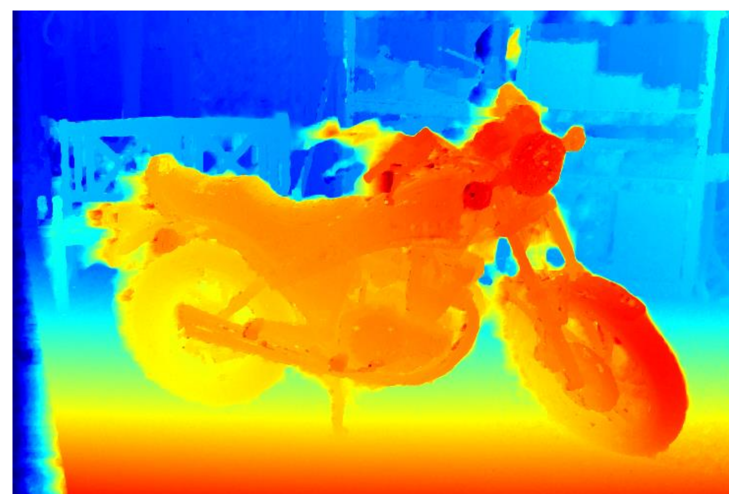
linear lifting

$L = 32$



precise lifting

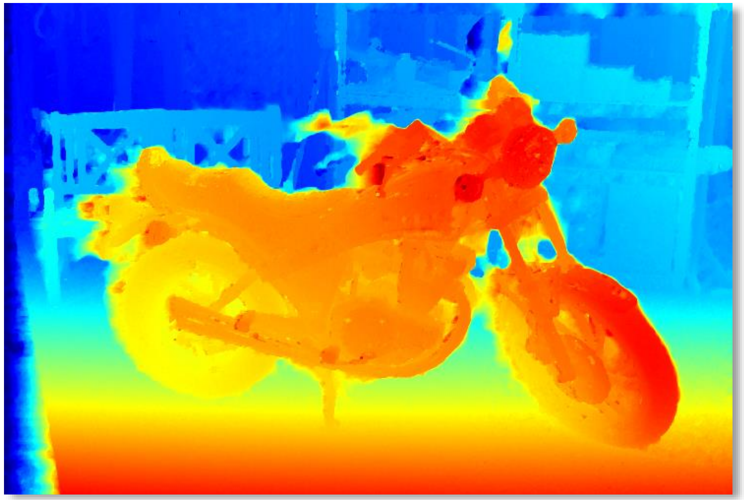
$L = 8$



Results – disparity estimation

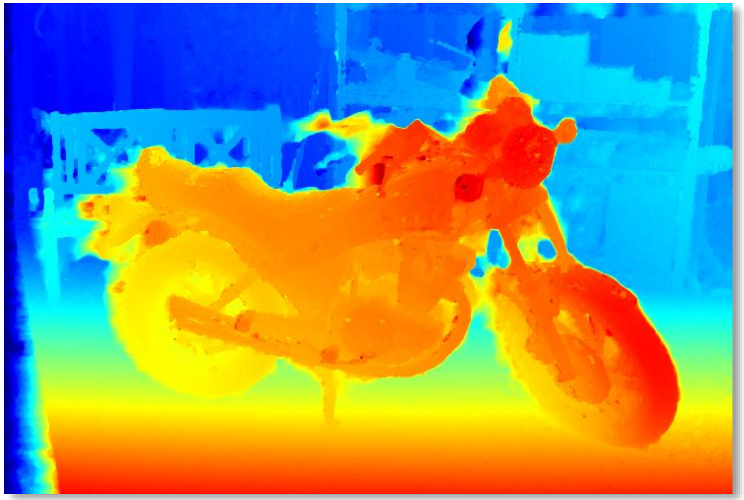
linear lifting

$L = 270$

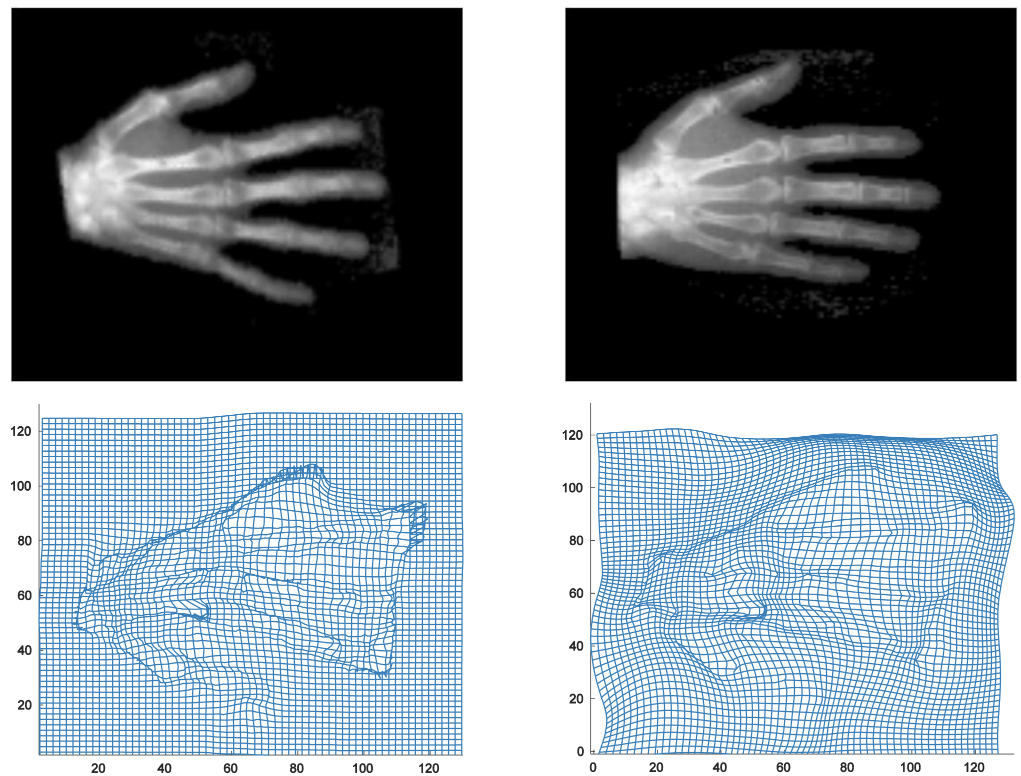


precise lifting

$L = 8$



Lifting in medical image registration



Goal: approximate global minimizers of

$$\int_\Omega f\big(x, u(x), D^2 u(x)\big)dx$$

Laplacian: $[p(x)]_{Lip(X)^d} \leq 1 \rightarrow [p(x)]_{Lip(X)^d} \leq 1$ and p concave [1,2]

[1] Löwenhauser, **Lellmann**'18; [2] Vogt, **Lellmann**'19

Take-home

- Measure-valued problems = nicer space & interesting math:
  - Wasserstein/Banach-TV (existence)
  - geometric data (discrete, manifolds)
  - approximating nonconvex problems

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