Relaxation Methods in Variational Image Processing

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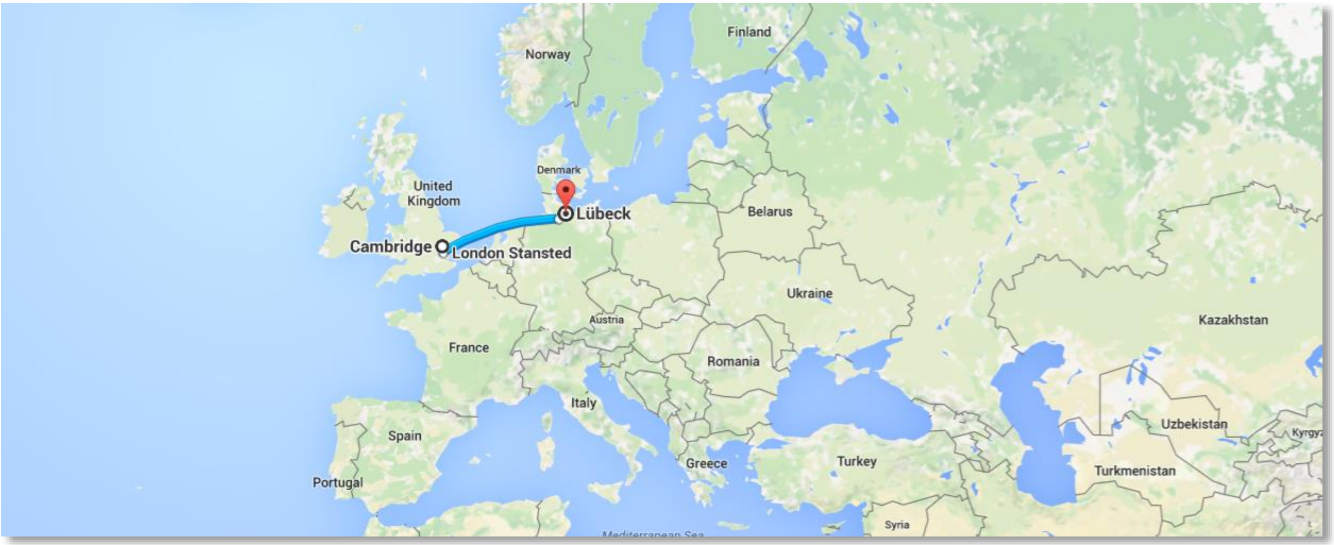
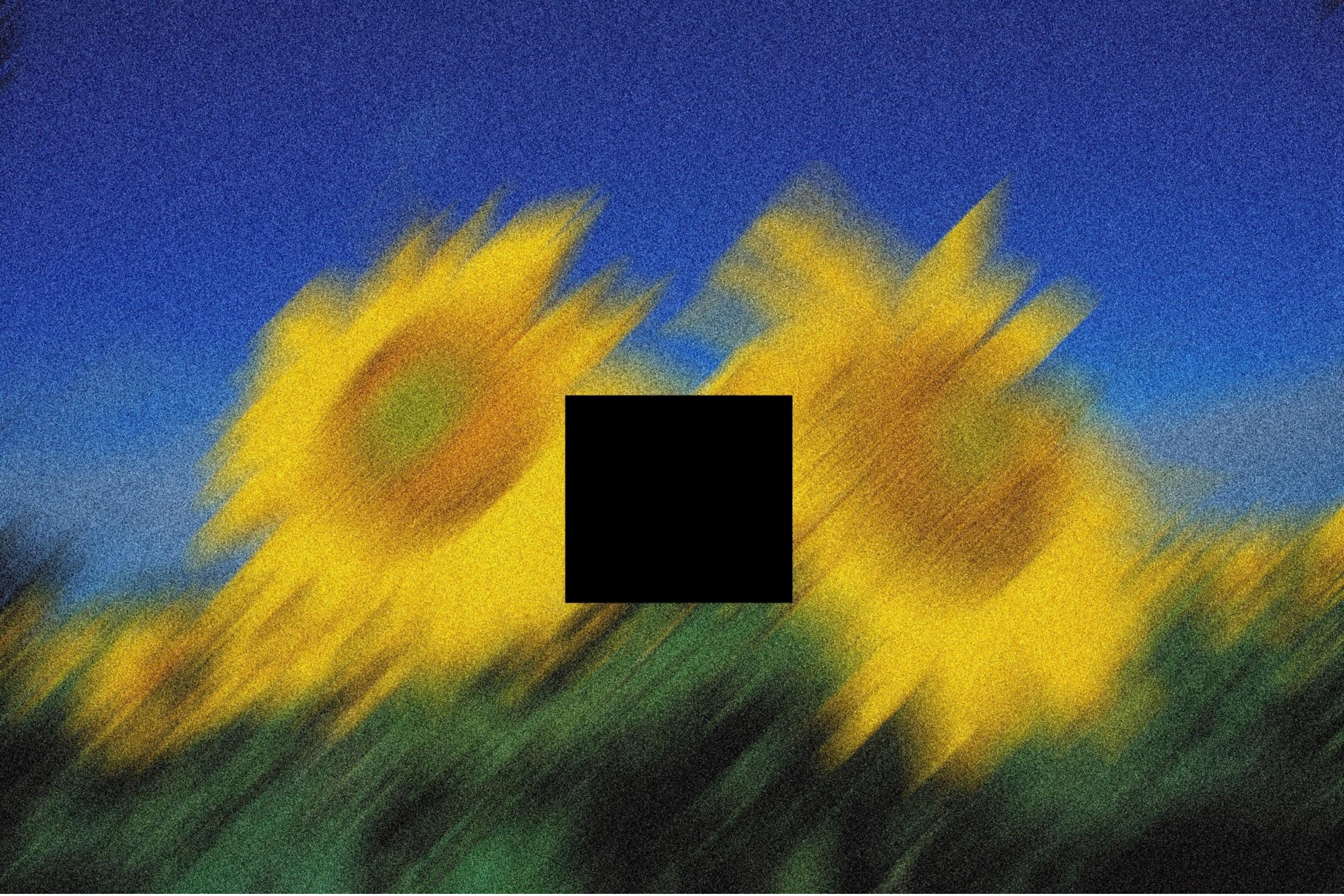


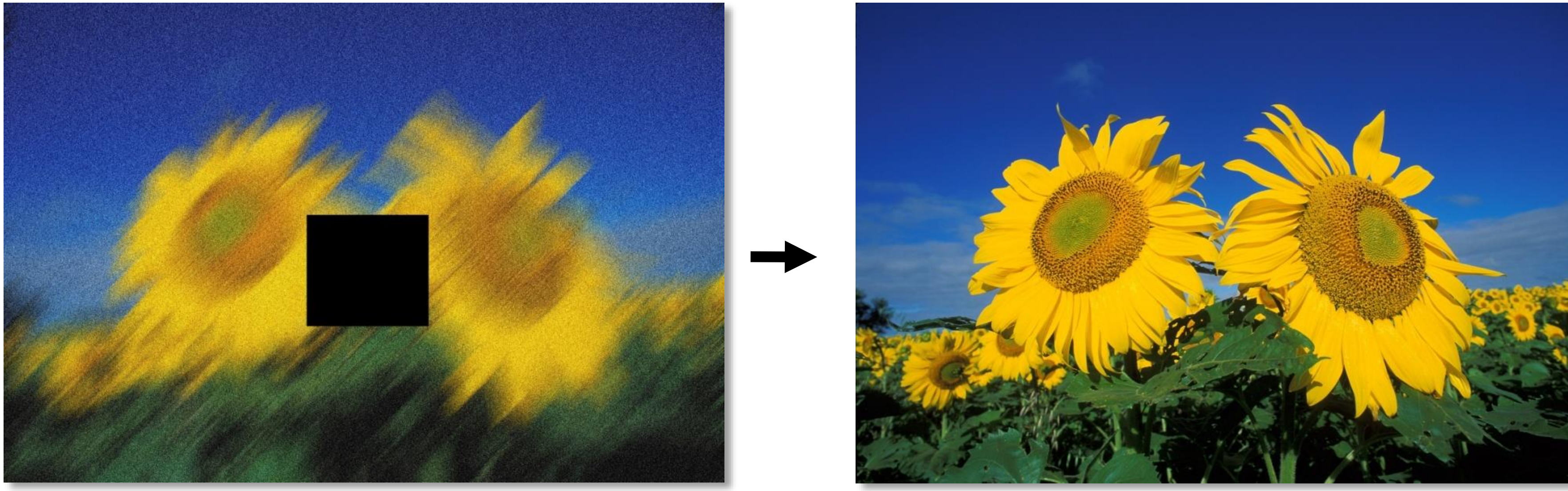
u

$b = T(u)$

$$b = T(u) + n$$



$$b = T'(u) + n'$$



Given measurements b, find image data u so that

$$b = T(u) + n$$

T structural operator, n random noise

Often the direct reconstruction is not unique, not stable, or not deterministic – we need prior knowledge

Variational methods

We reconstruct the unknown data u from the measurements b by minimizing the energy

$$\min_u \{D(T(u); b) + R(u)\}$$

Intuitive (what do we want) and modular (reusable)

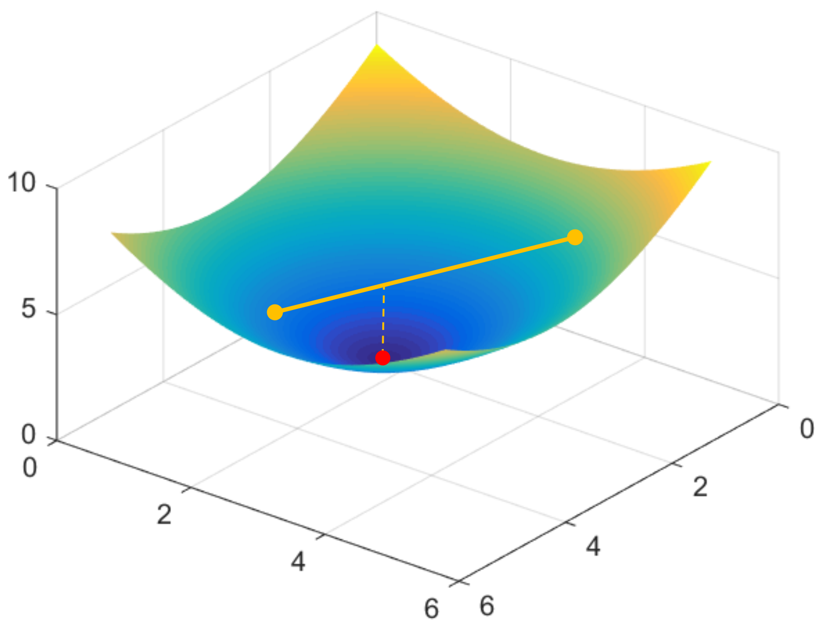
In practice: often

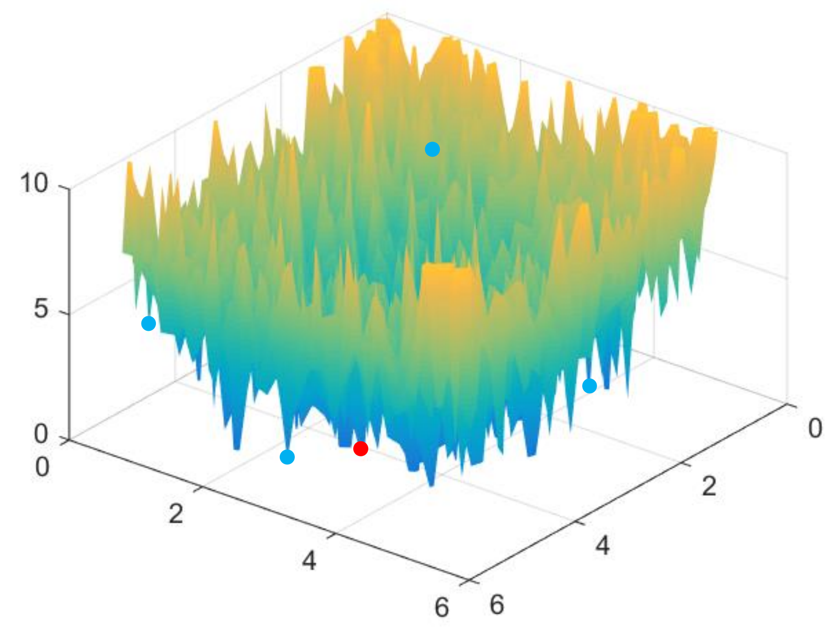
$$\min_{u:\Omega\to X} \int_\Omega \rho(x, u(x))dx + \lambda \int_\Omega \sigma(\nabla u)dx$$

Tikhonov '63

Convexity

Convexity assures that every local minimizer of the energy is also a global minimizer





Non-convexity I – data terms

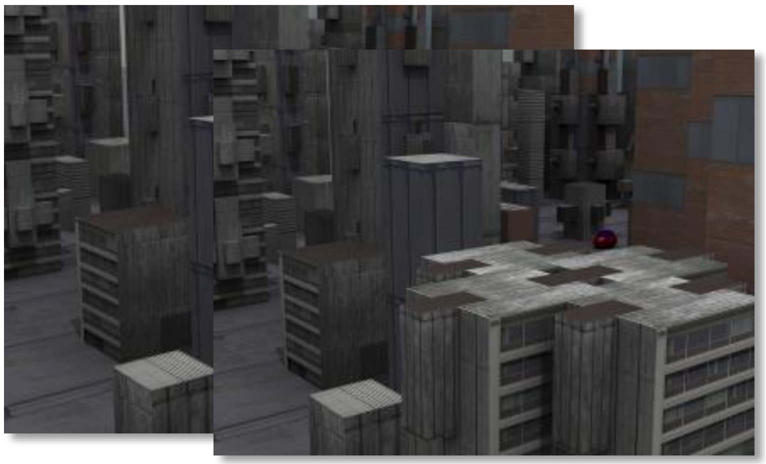


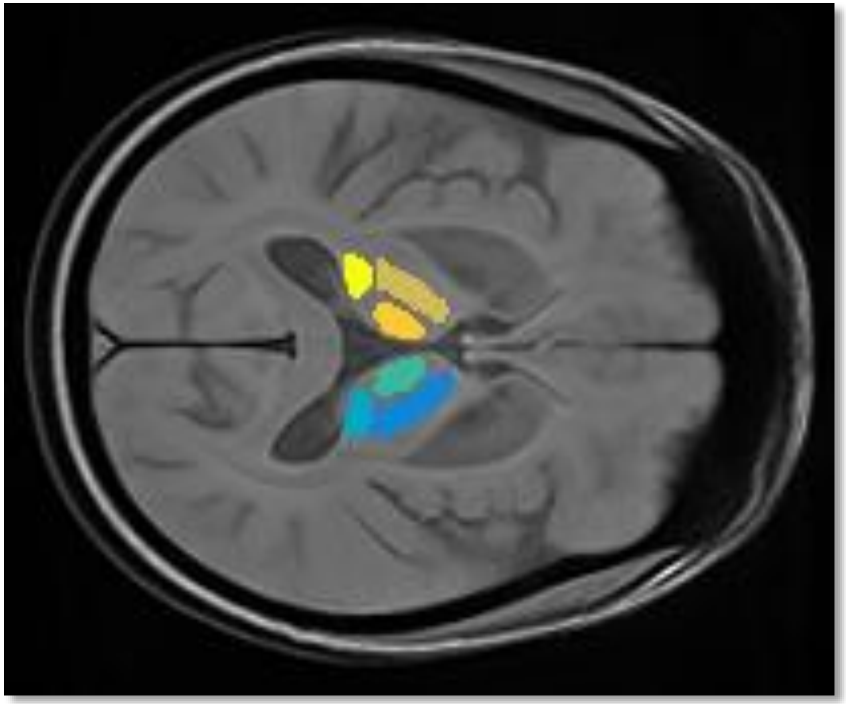
image sequence

optical flow/registration

$$\min_{u:\Omega\to\mathbb{R}^2} \int_\Omega D(I_\sigma^1(x), I_\sigma^2(x+u))dx + \lambda \int_\Omega d\|Du\|$$

Lellmann, Strekalovskiy, Kötter, Cremers '13

Non-convexity II – labeling

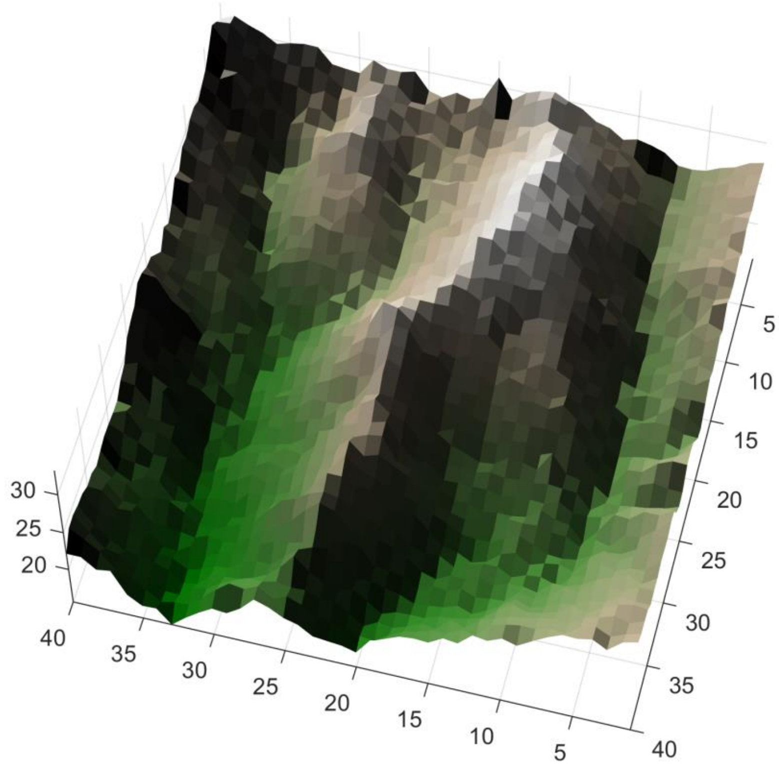


$$\min_{u:\Omega\to X} f(u) := D(u; I) + R(u)$$

$$X := \left\{ \blacksquare \quad \blacksquare \quad \blacksquare \quad \dots \right\}$$

With: V. Corona, C. Schönlieb, J. Acosta-Cabronero, P.~Nestor/DZNE Magdeburg

Non-convexity III – manifolds



$$\min_{u:\Omega\to\mathcal{M}} \int_{\Omega} D_{\mathcal{M}}(u(x), I(x))dx + \lambda\,\mathrm{TV}(u)$$

Cremers, Strekalovskiy '12
Lellmann, Strekalovskiy, Kötter, Cremers '13

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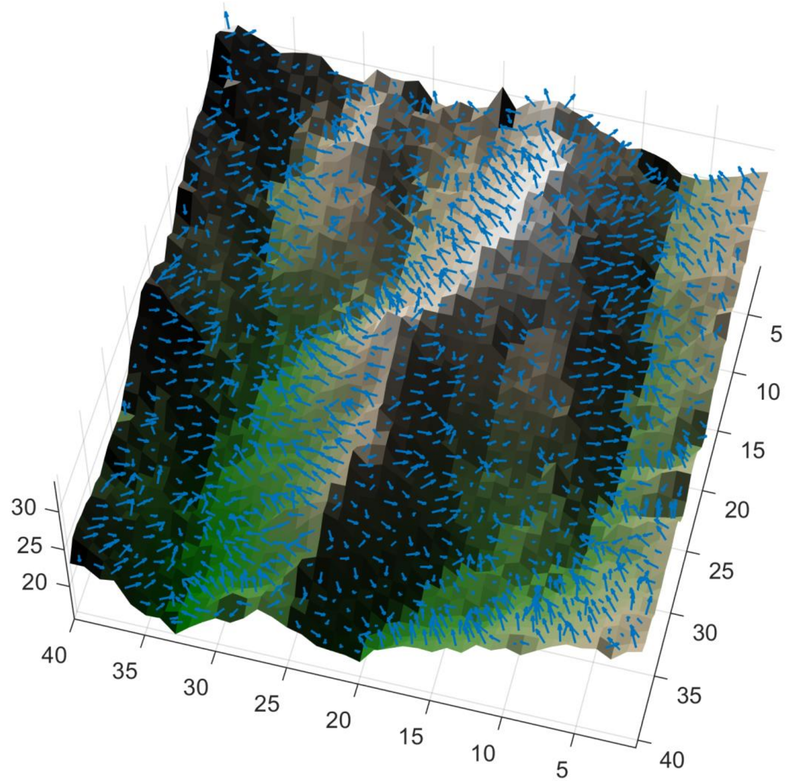

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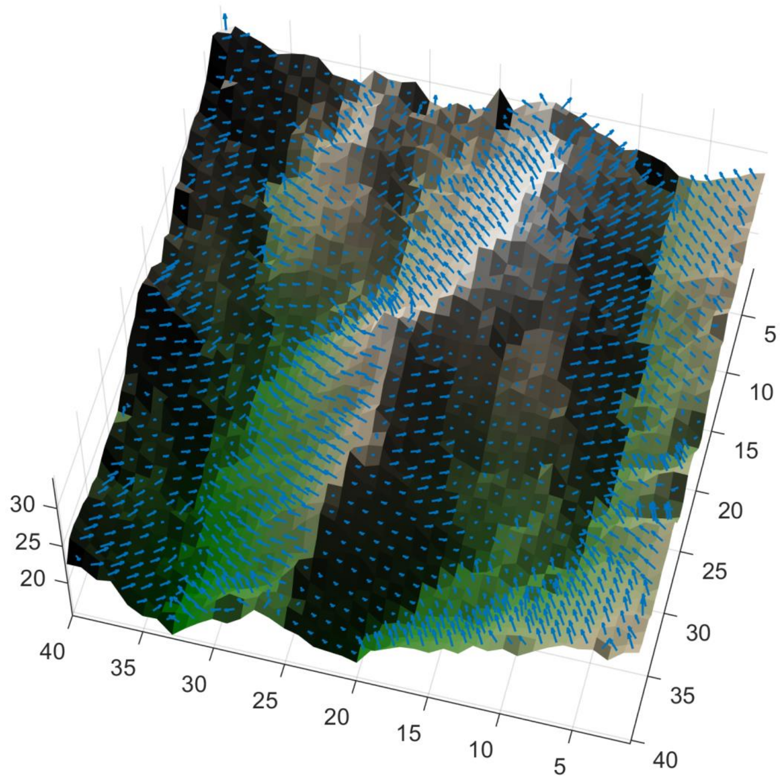

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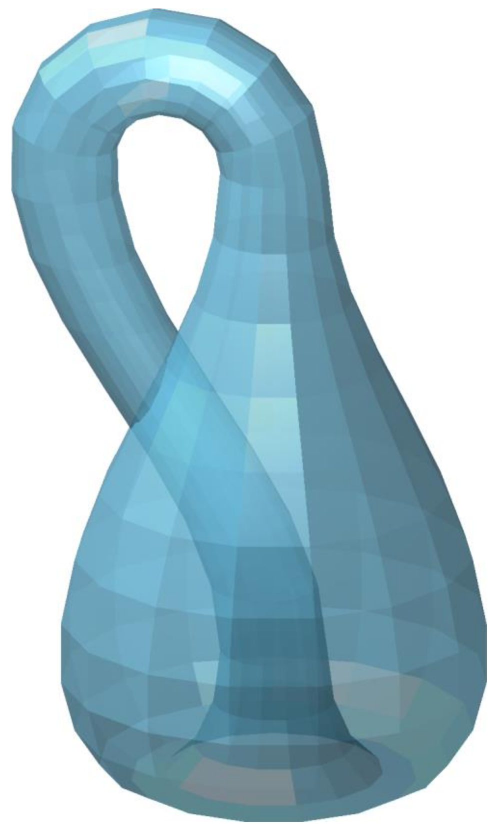

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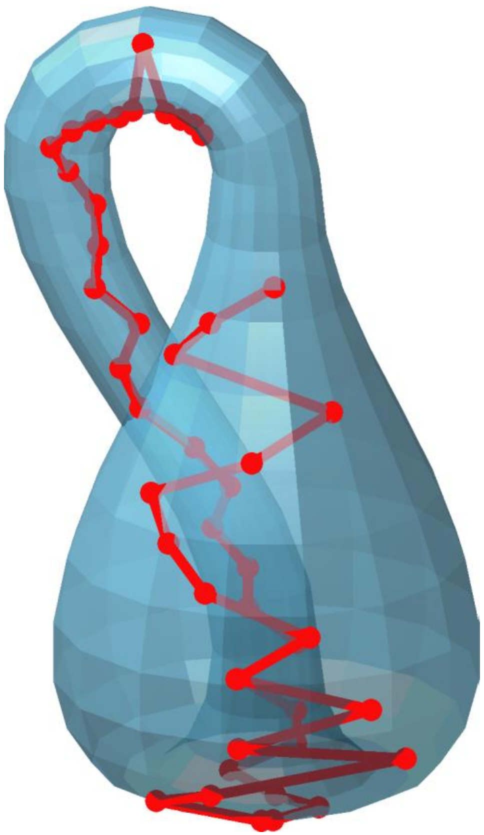

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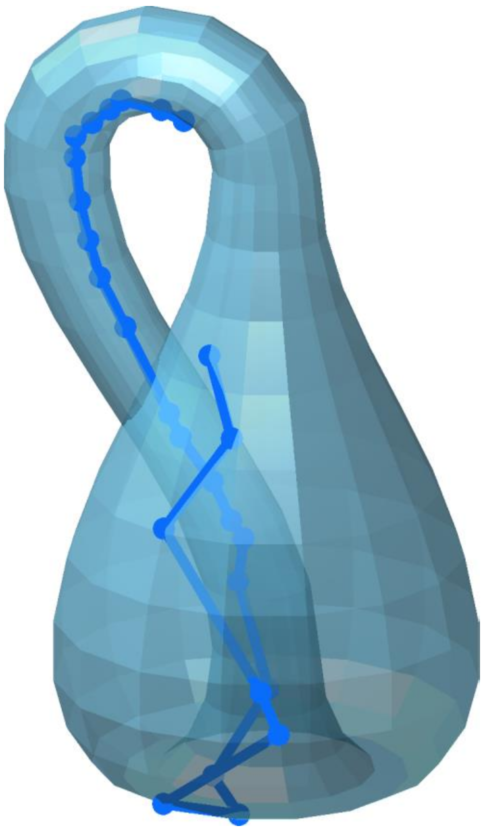

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Exact convex relaxation

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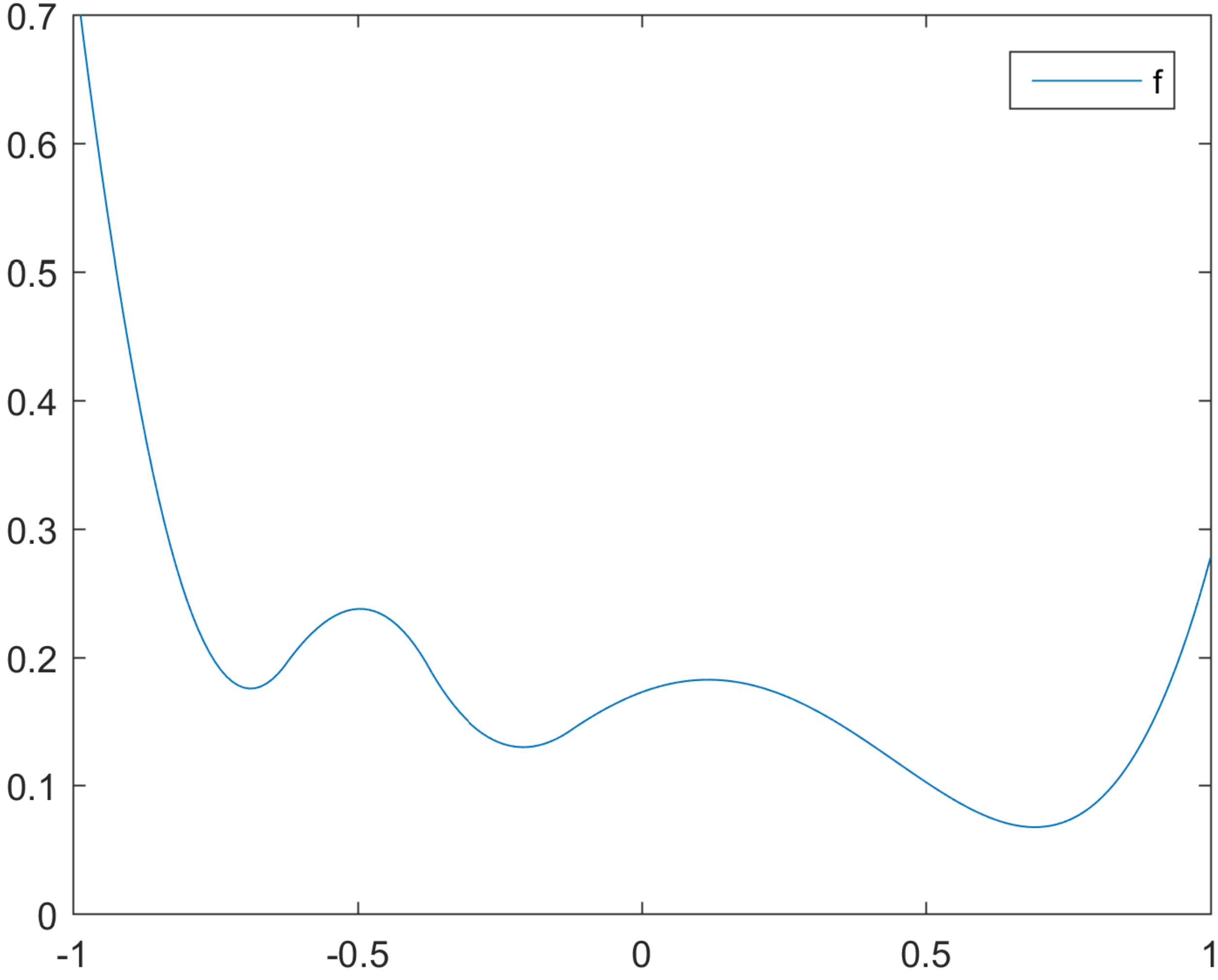
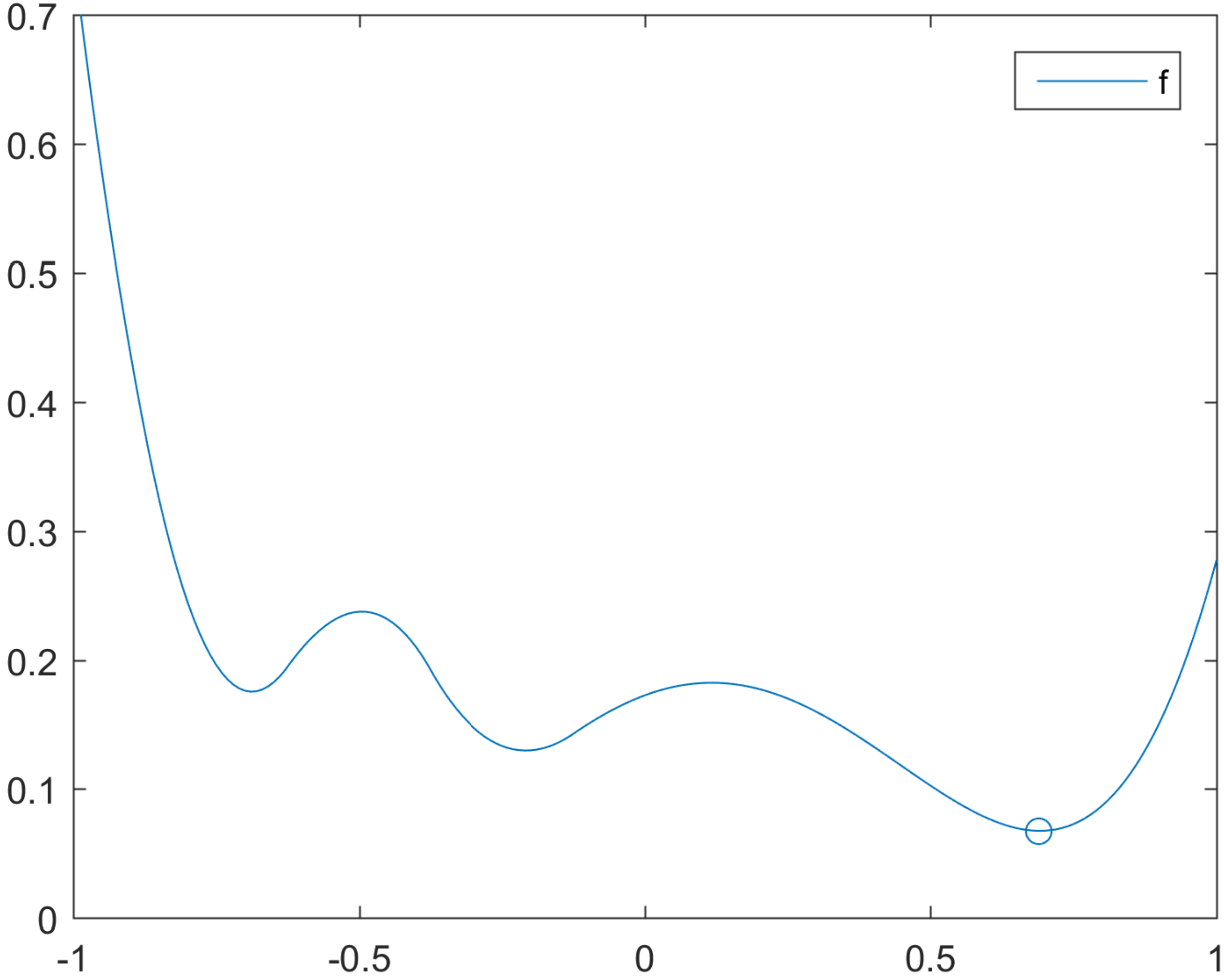


Exact convex relaxation



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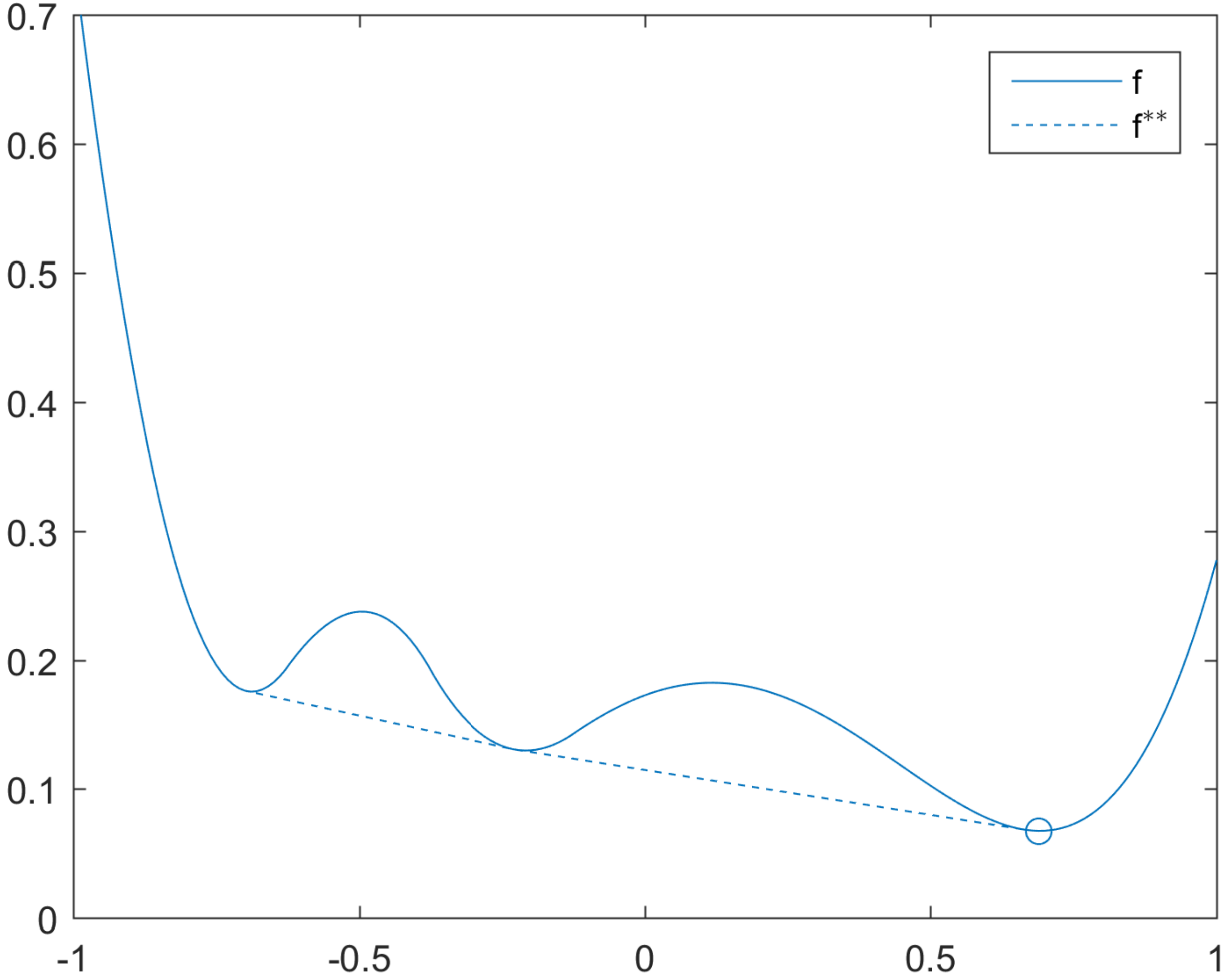

Drawback

This is generally as hard as minimizing the original energy!

Objectives are often sums

$$\min_{u:\Omega\to X} \int_\Omega \rho(x, u(x))dx + \lambda \int_\Omega \sigma(\nabla u)dx$$

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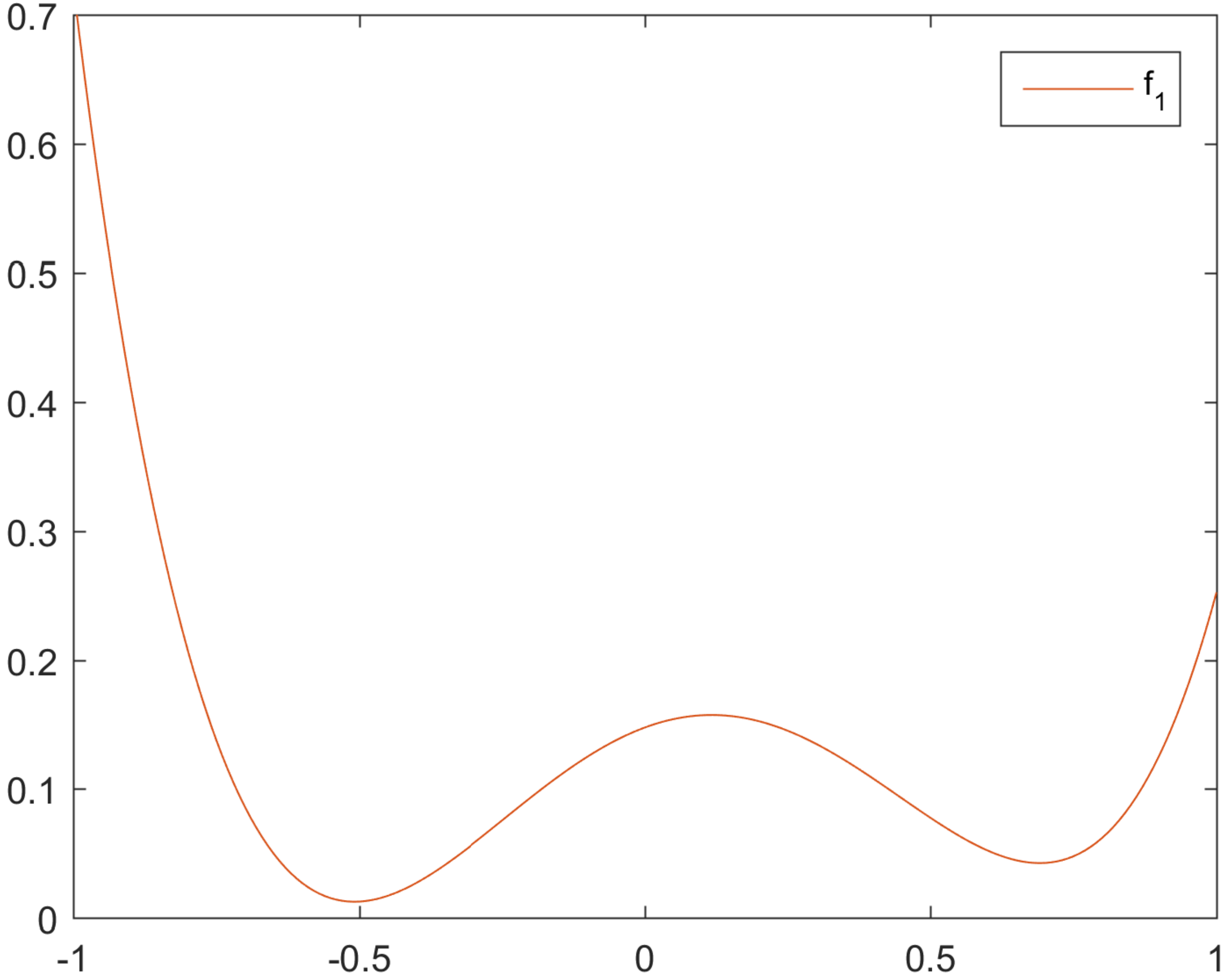

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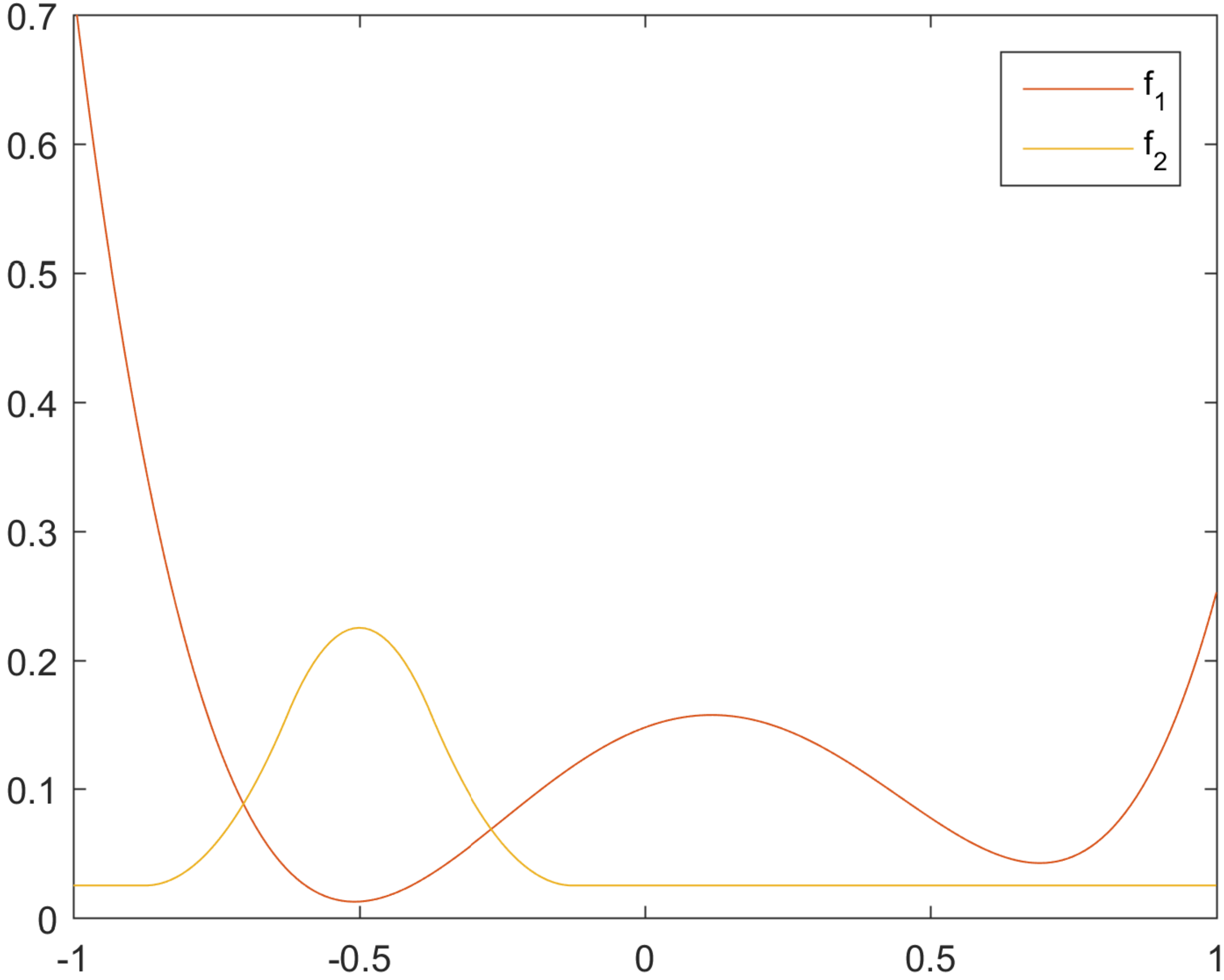

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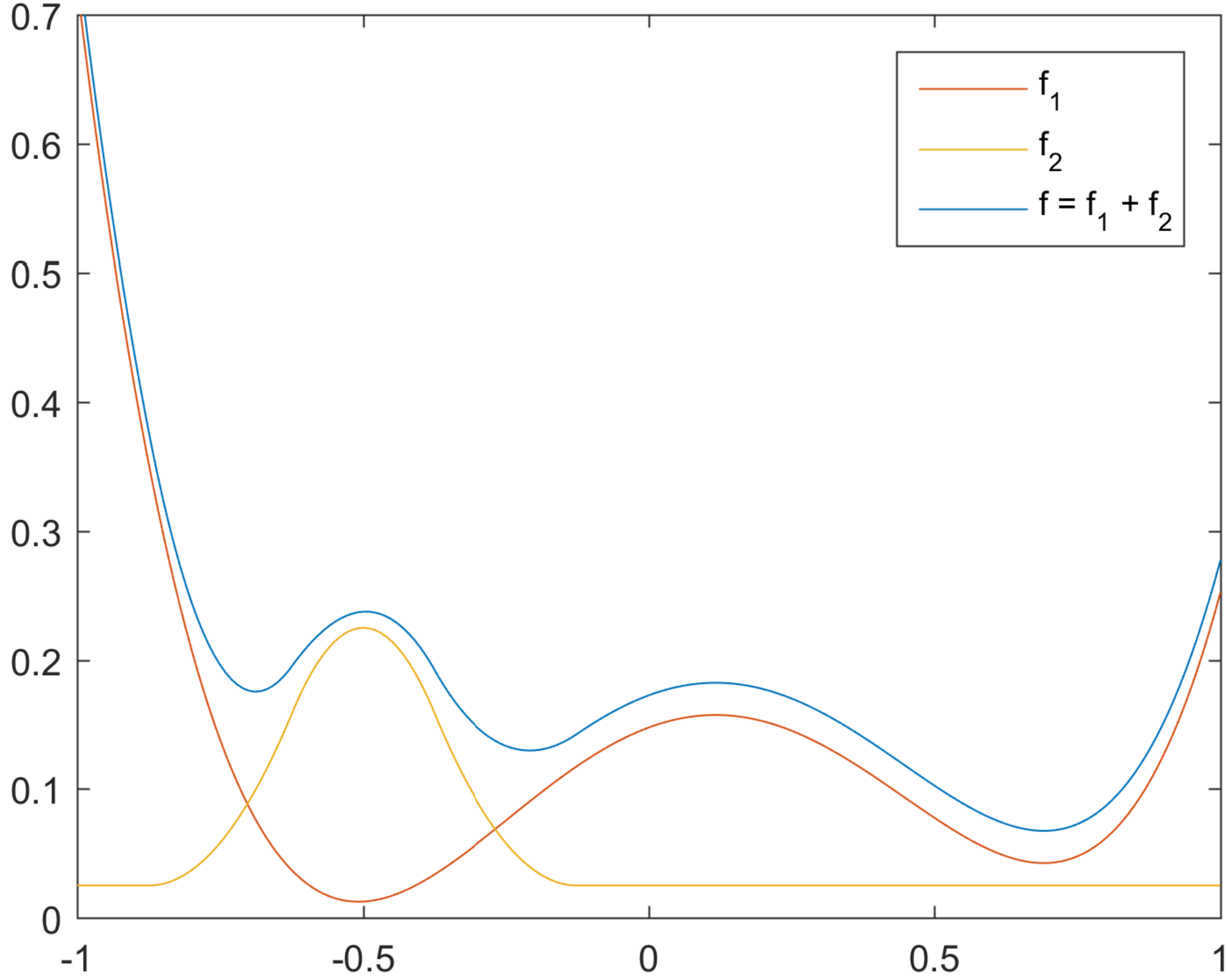

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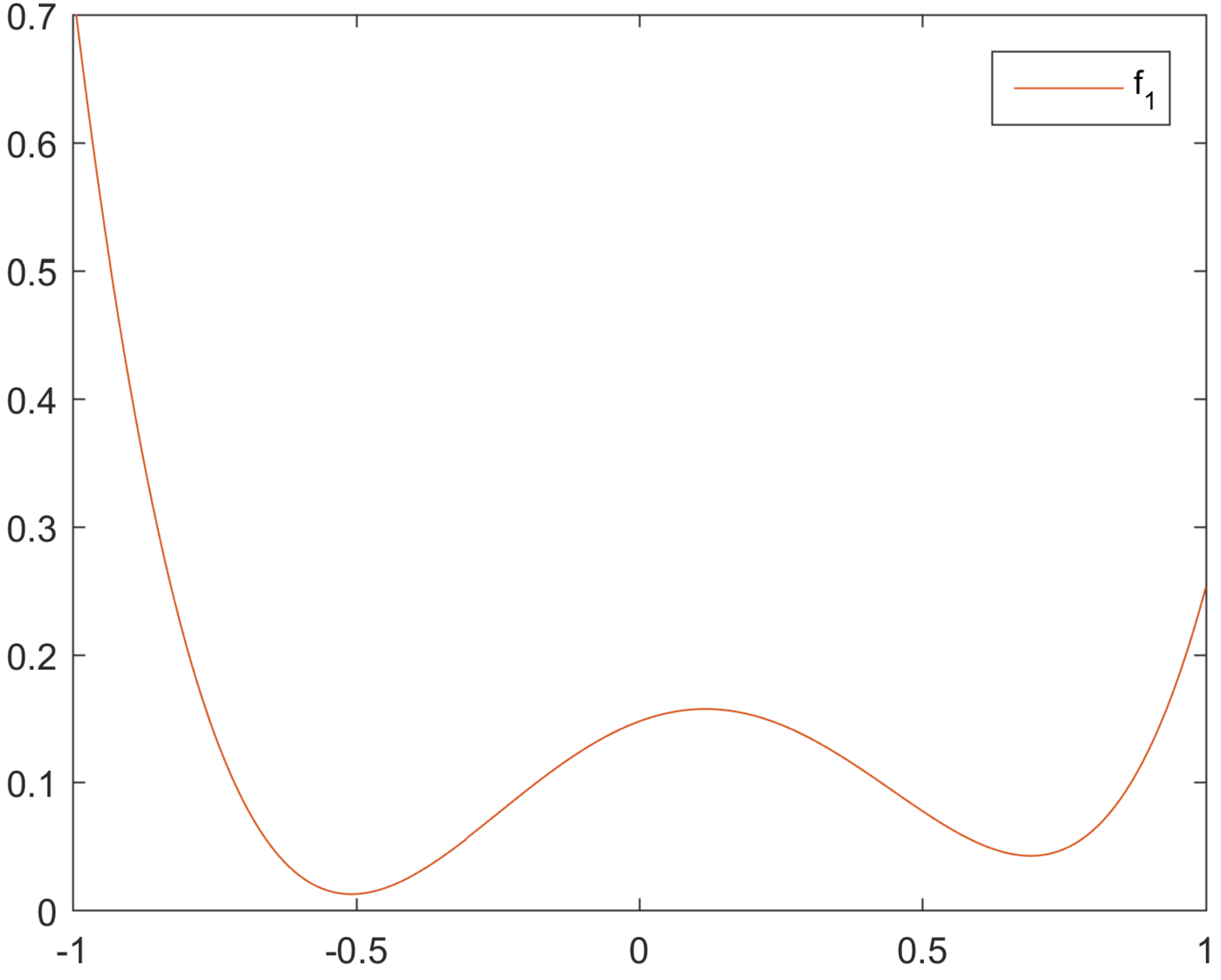

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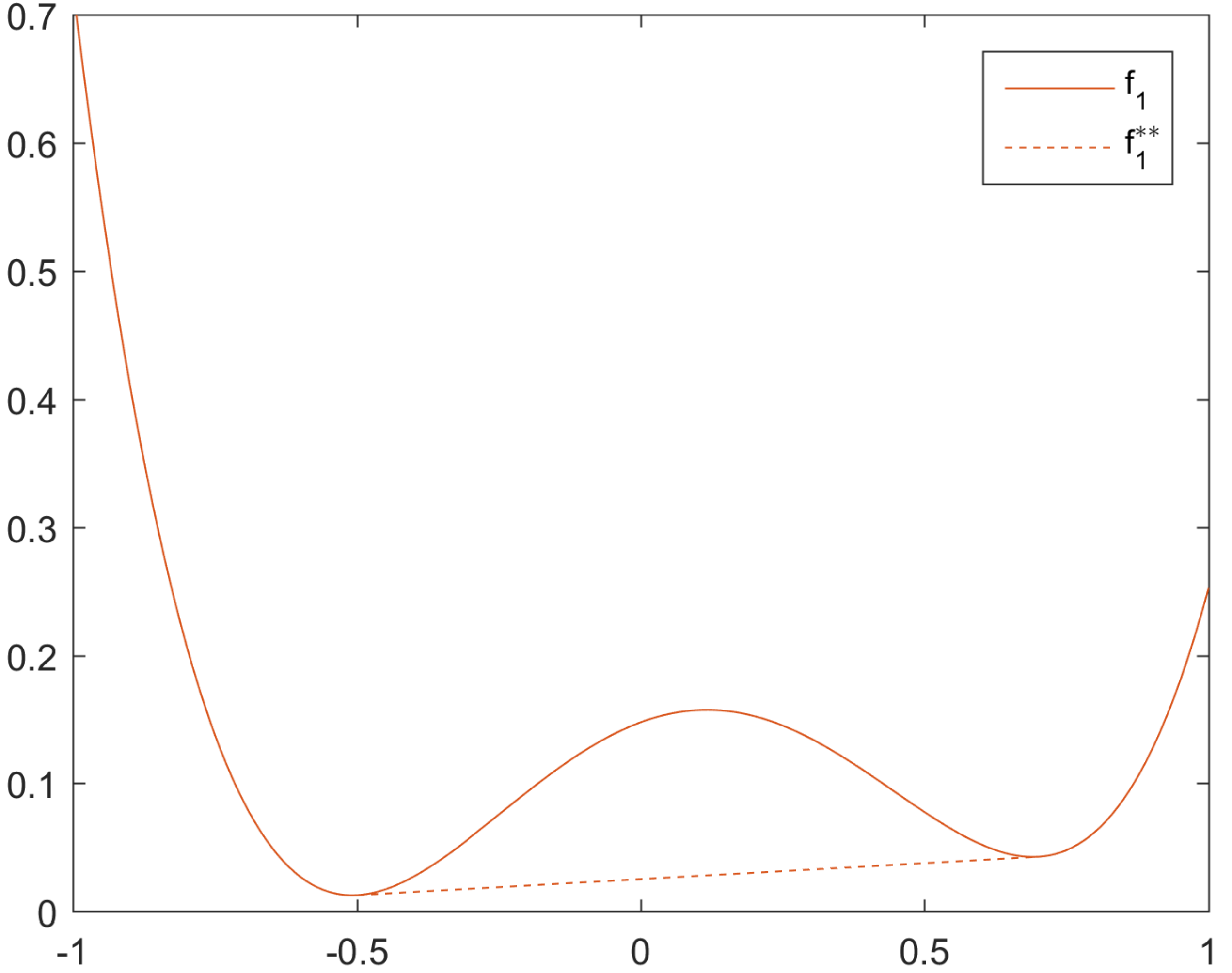

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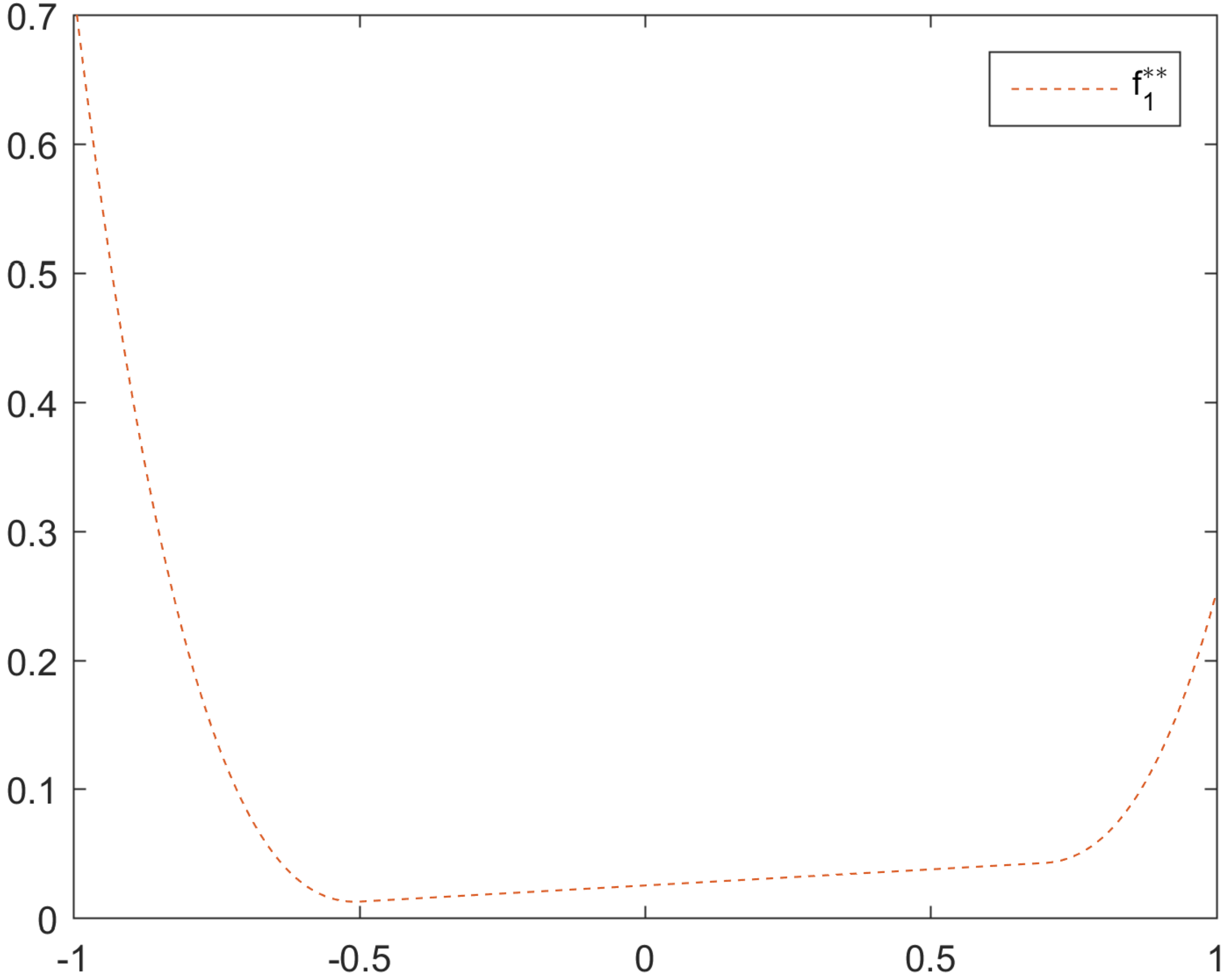

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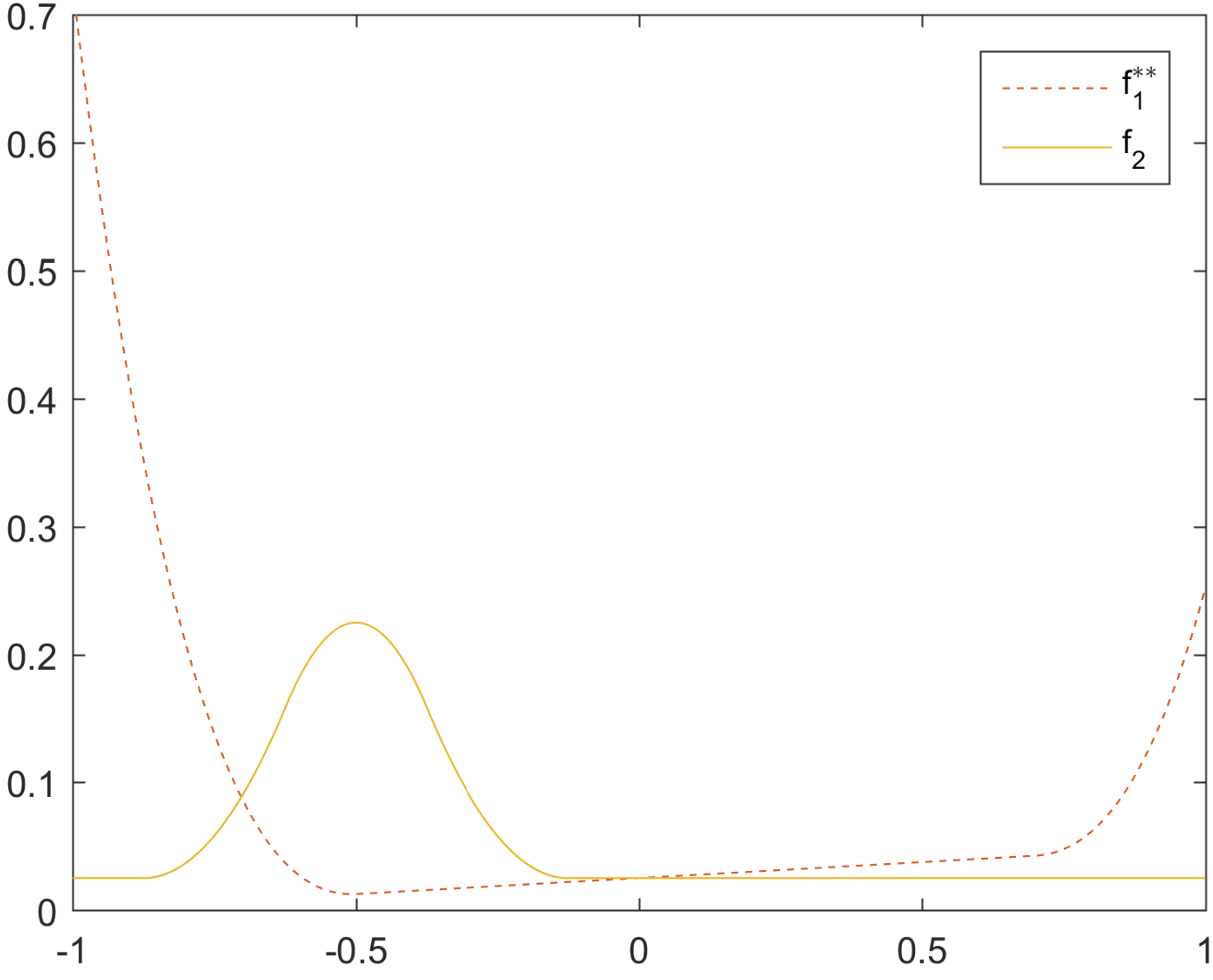

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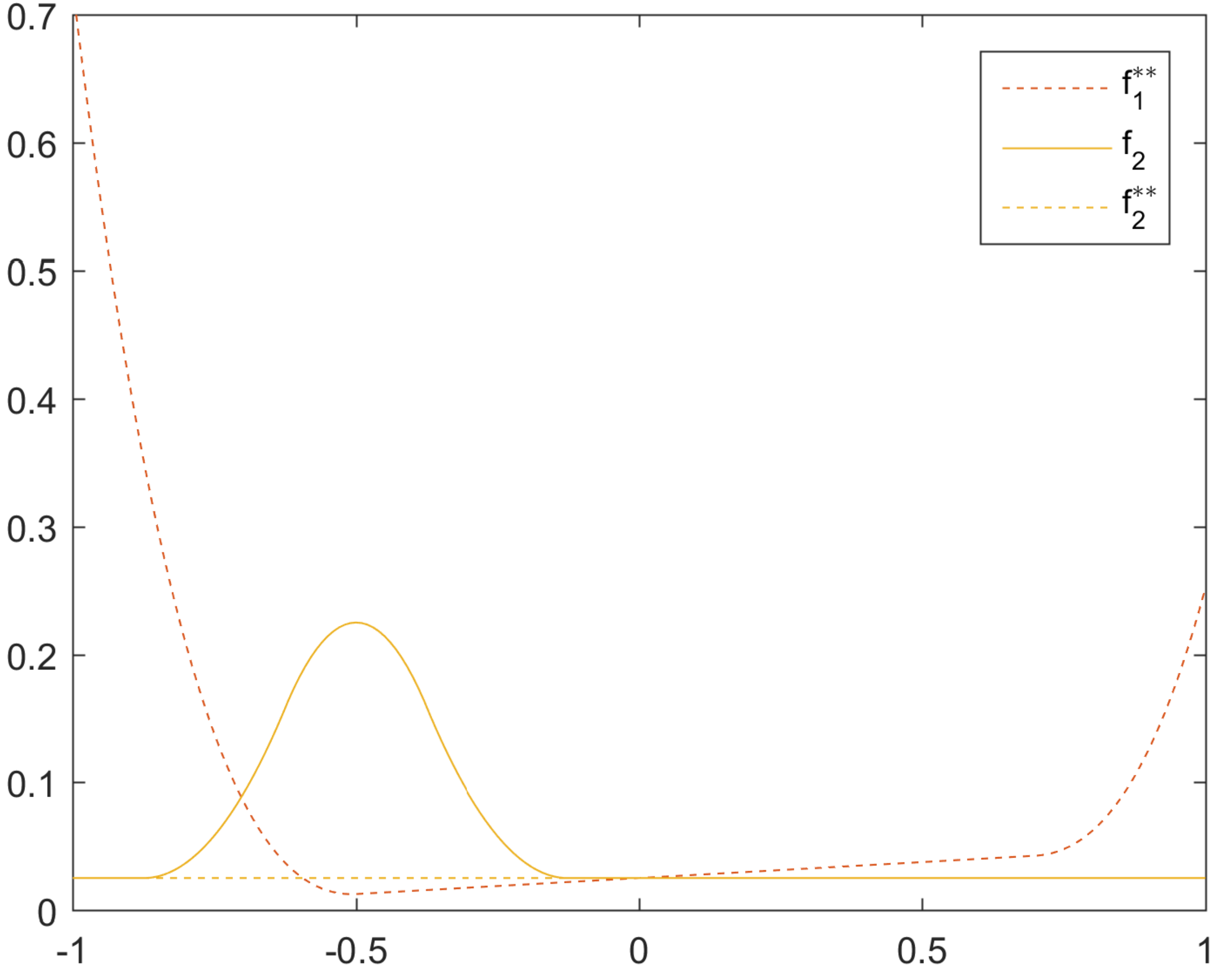

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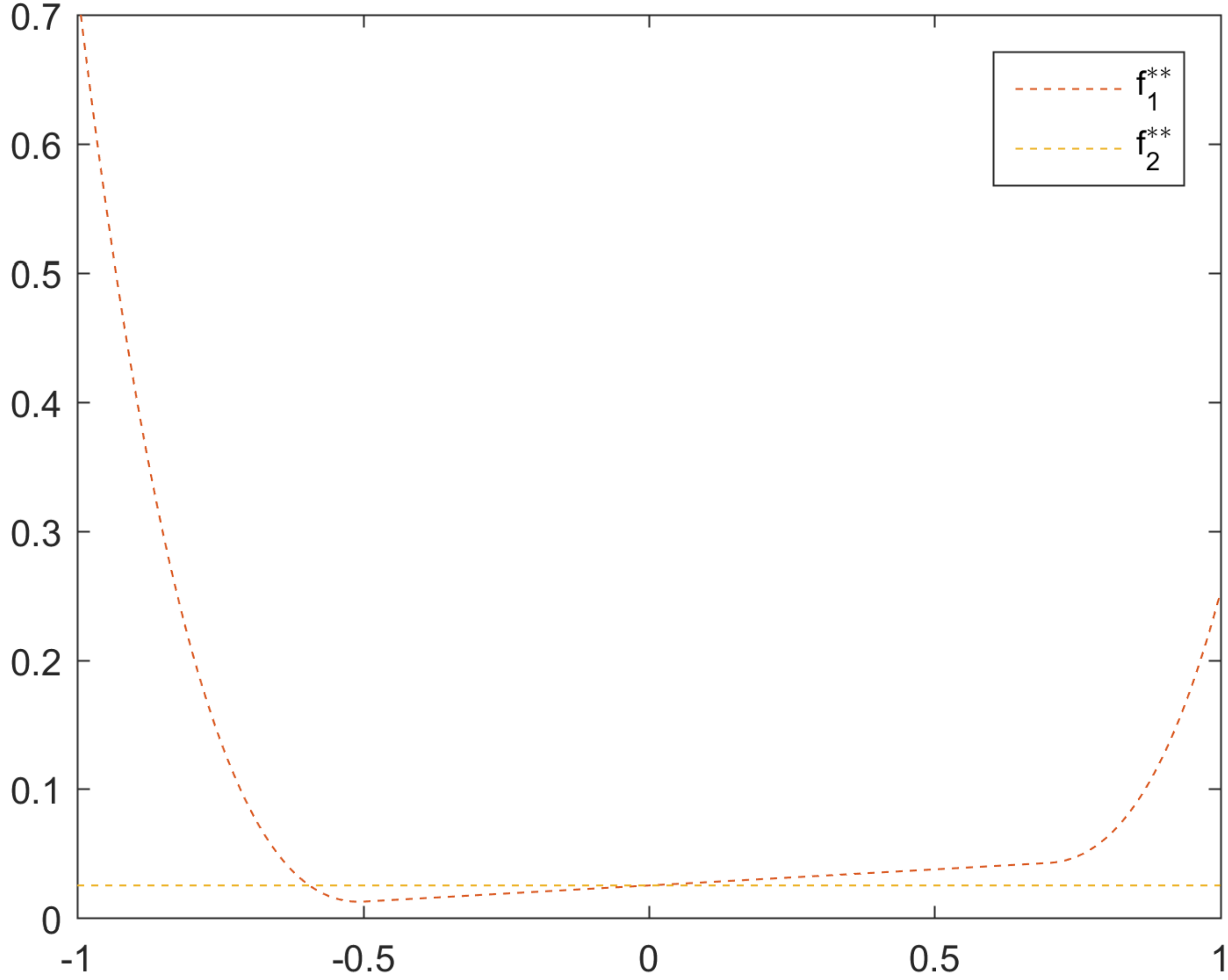

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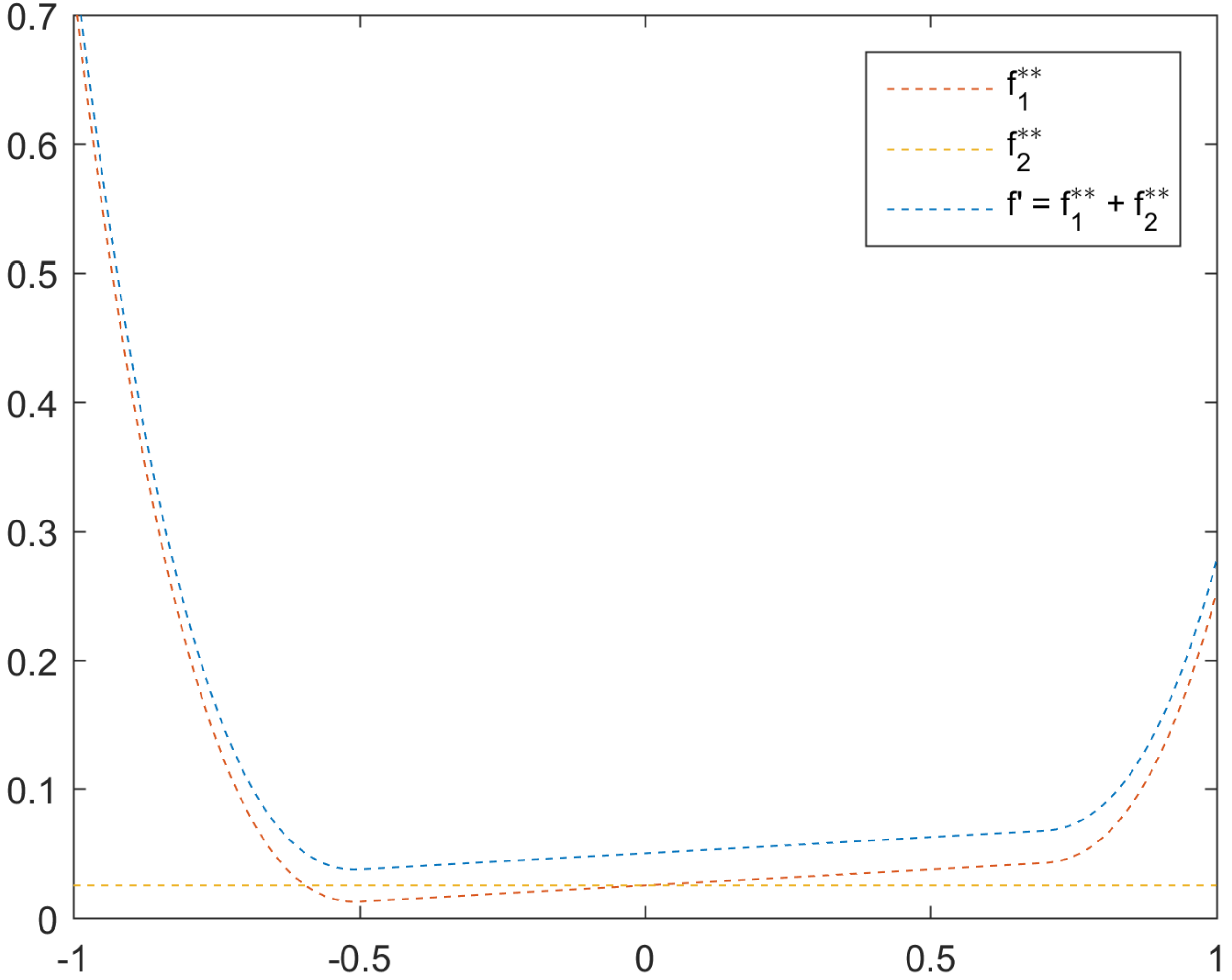

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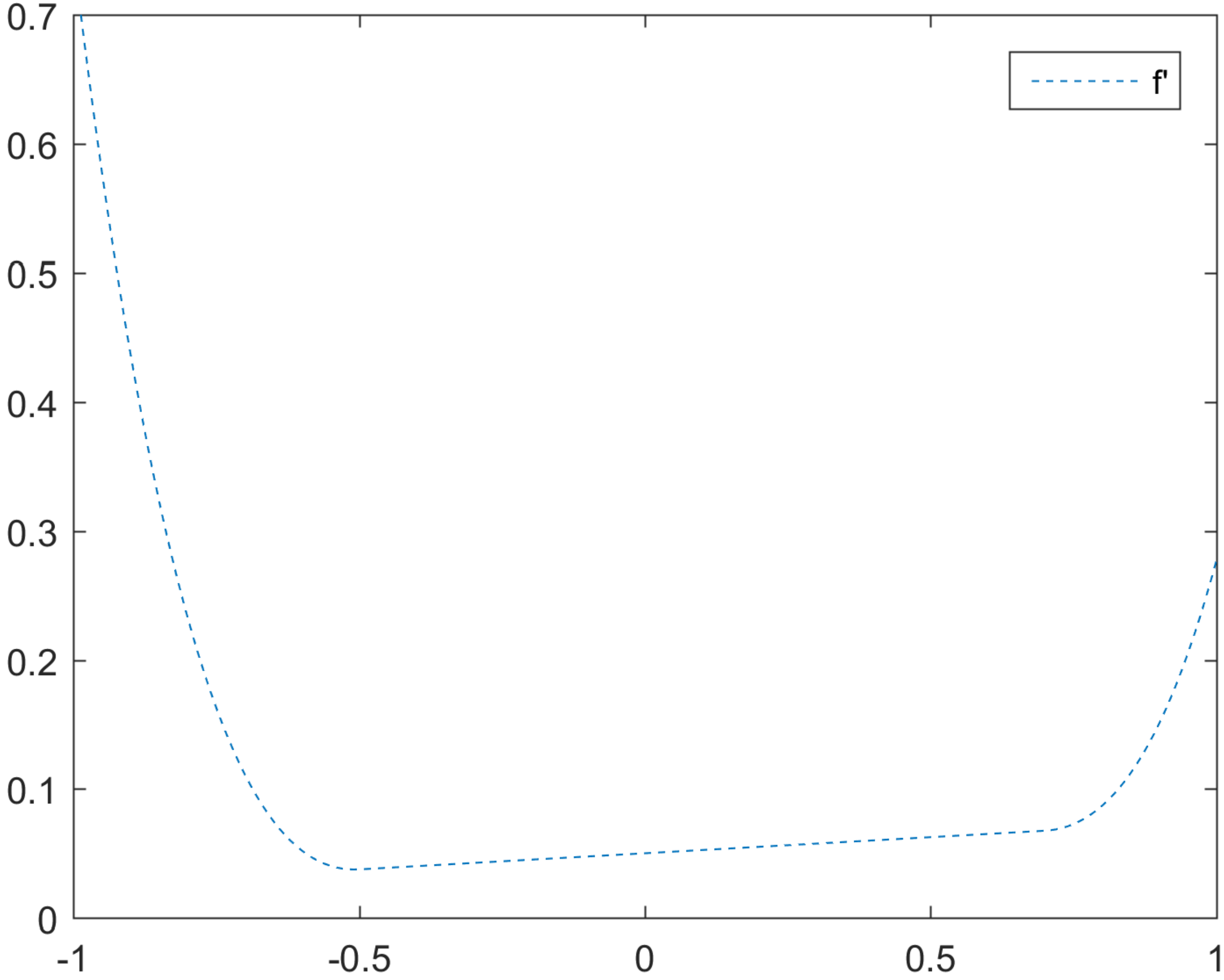

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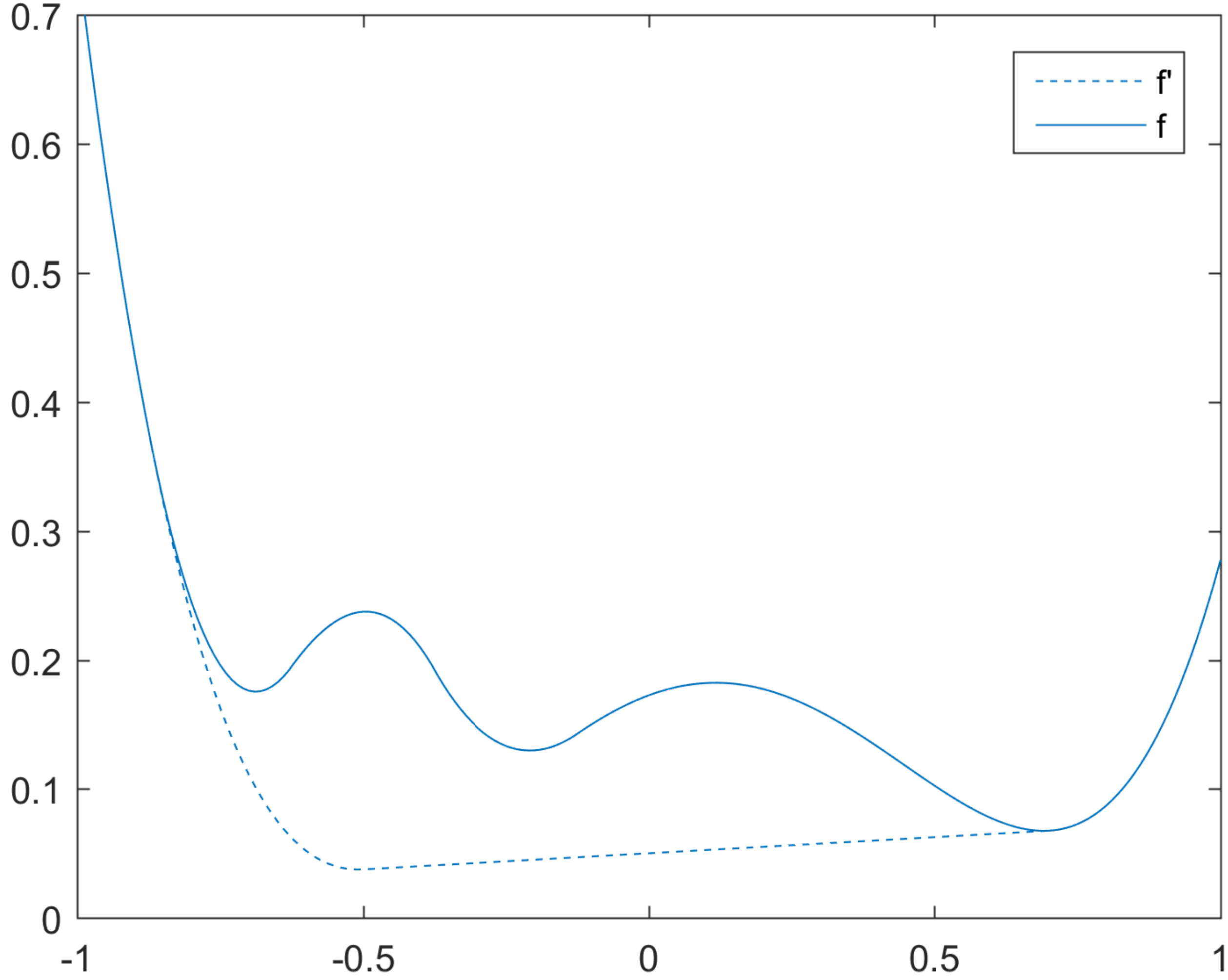

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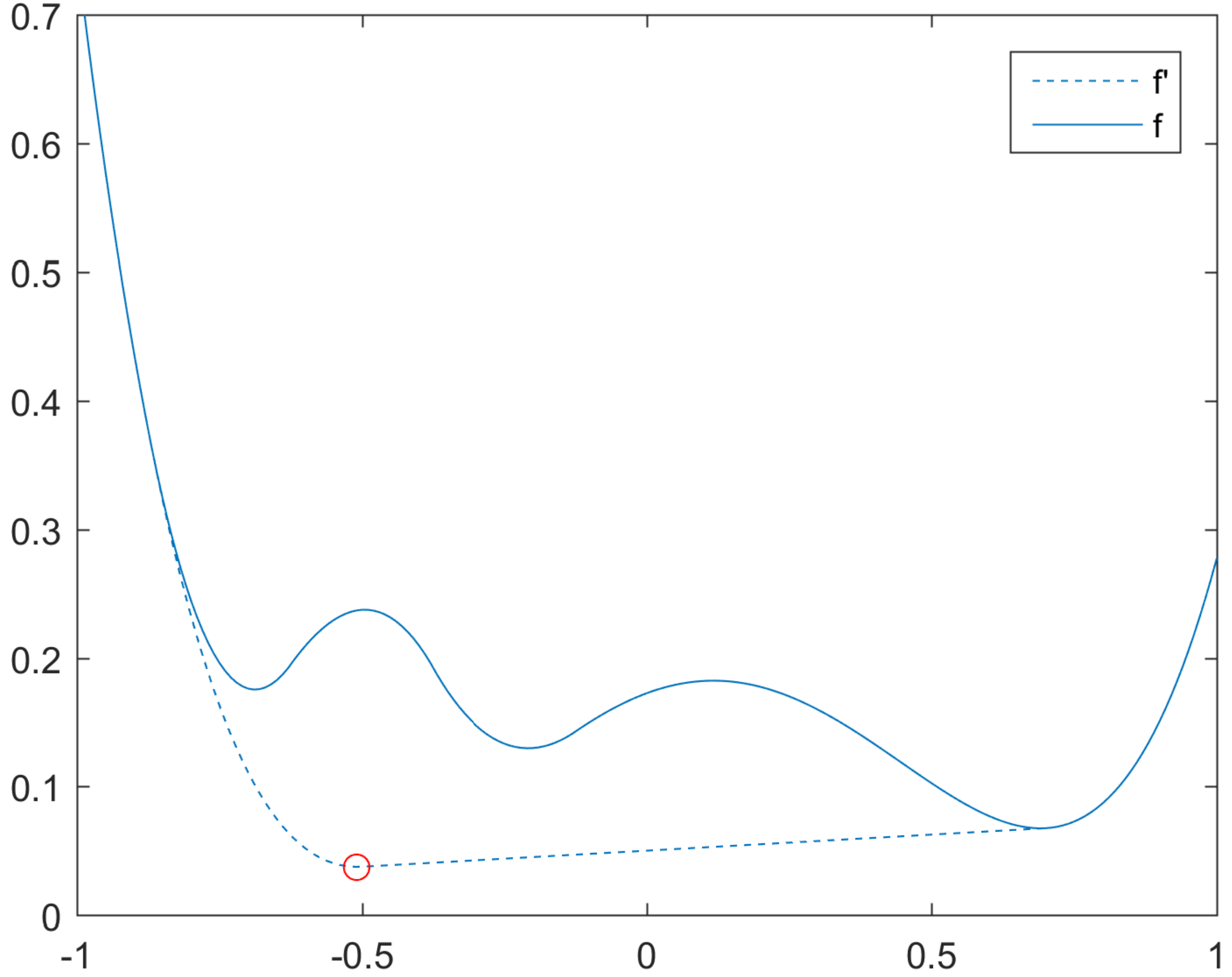
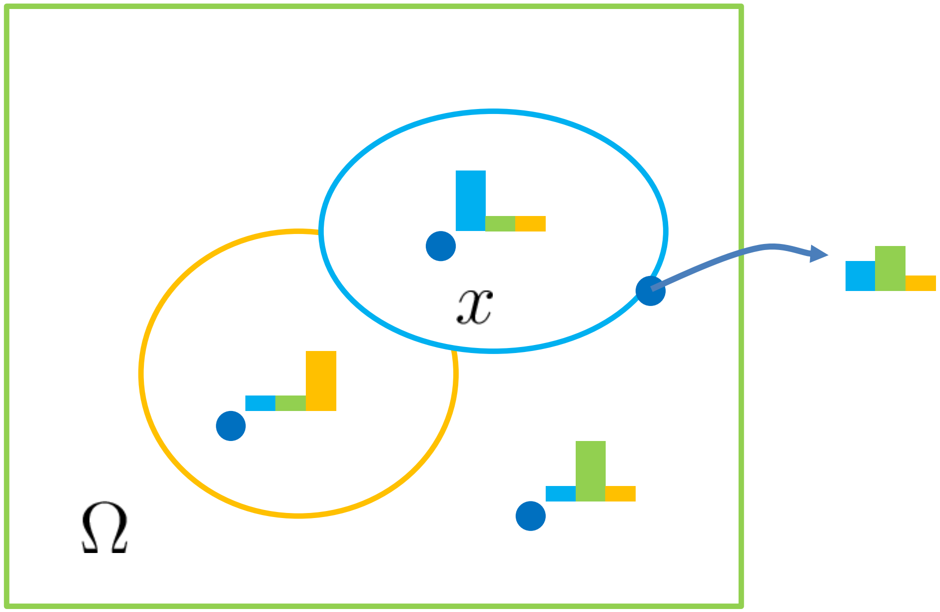


Lifting



Hard decisions are replaced by soft "probabilities"

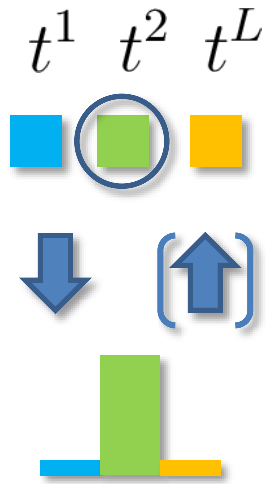
Lifting

We extend the problem

$$\min_{u':\Omega\to X} f'(u')$$

to the probability measures:

$$\min_{u:\Omega\to\mathcal{P}(X)} f(u)$$



The new energy f should agree with f' on Dirac measures, and not create artificial minimizers.

Potts '52; **Boykov** et al. '98, '01; Kleinberg, **Tardos** '01
Zach et al. '08; Lellmann, Becker, Schnörr '09; **Chambolle**, **Cremers**, Pock '11
Young measures: Young '37; Currents: Schwartz '51; de Rahm '55; Federer '69
Paired calibrations: Brakke '91; **Alberti, Bouchitté, Dal Maso** '01

Linear relaxation

- Lifting + relaxation using the biconjugate:

$$\int_\Omega \rho(u(x))dx \rightsquigarrow \int_\Omega \boldsymbol{\rho}^{**}(u(x))dx$$

Möllenhoff, **Laude**, **Möller**, **Lellmann**, Cremers, CVPR'16
related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

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- Linear relaxation (*1-sparse* solutions):

$$\boldsymbol{\rho}(z) = \begin{cases} \rho(t^i), & z = e^i,\ i \in \{1, \ldots, L\}, \\ +\infty, & \text{otherwise.} \end{cases}$$

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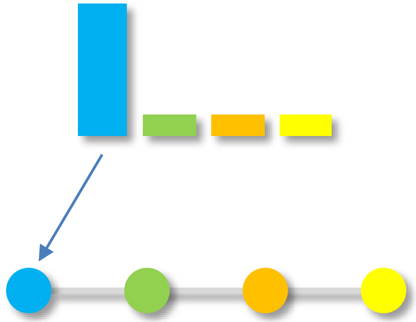

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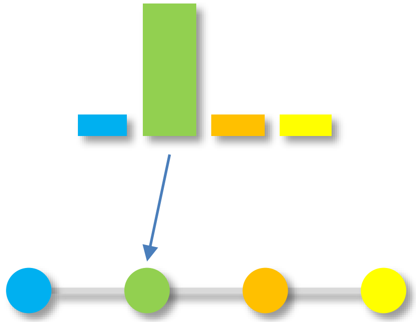

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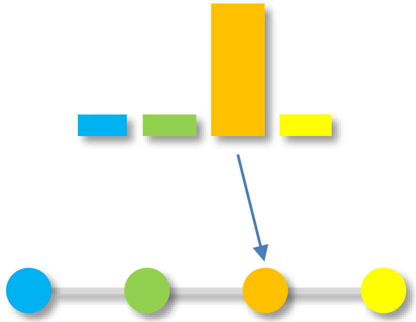

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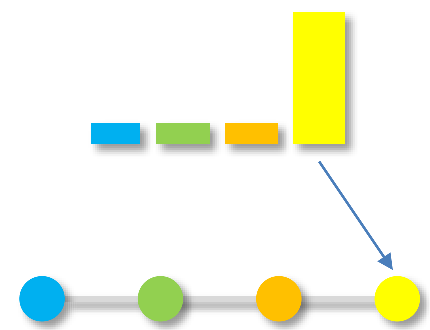

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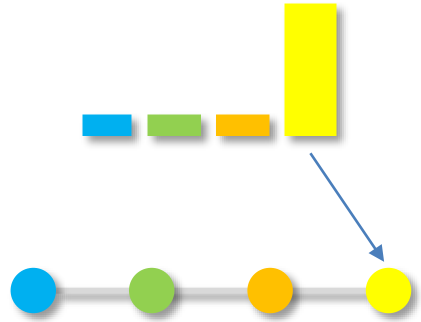

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Lifting + linear relaxation

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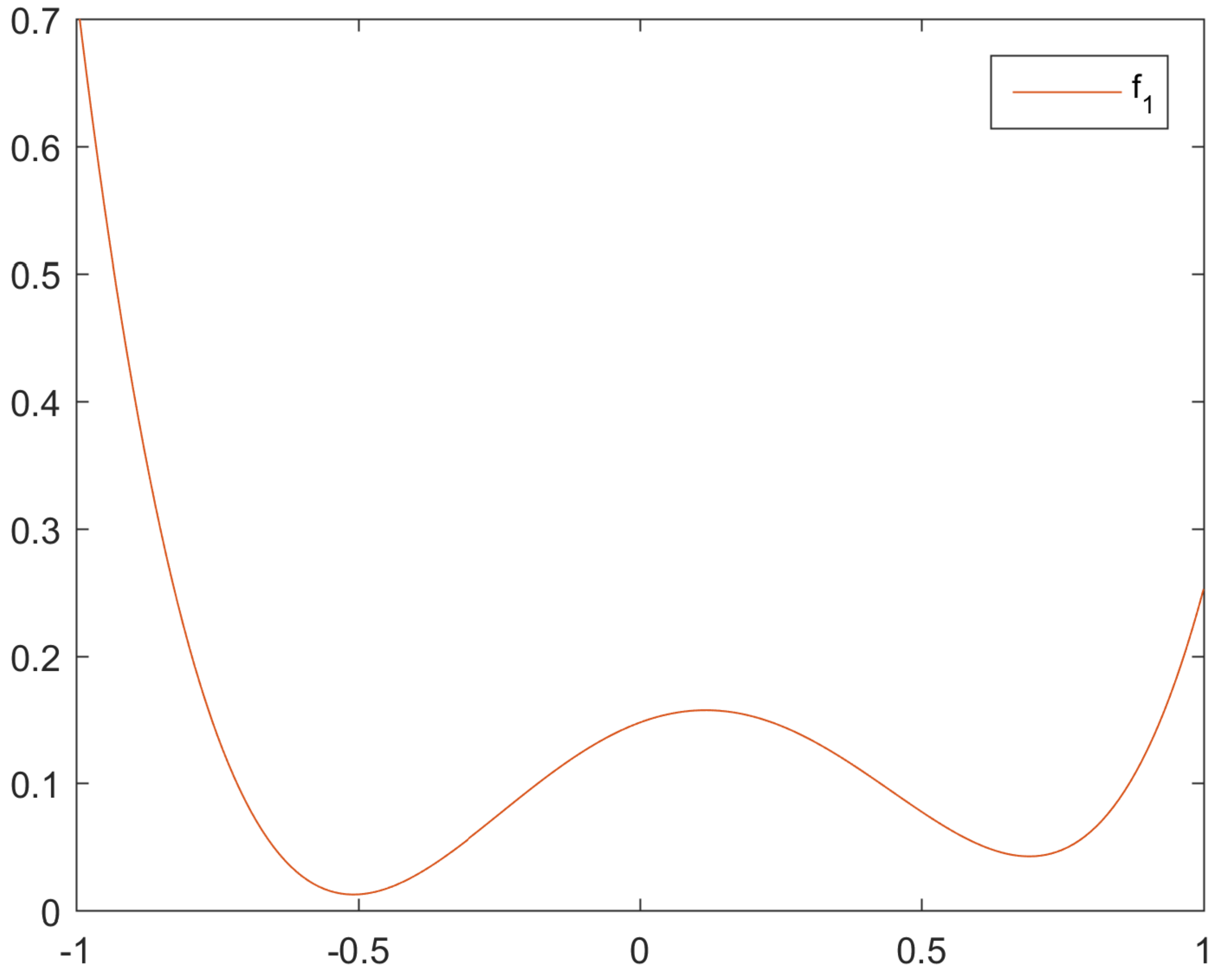

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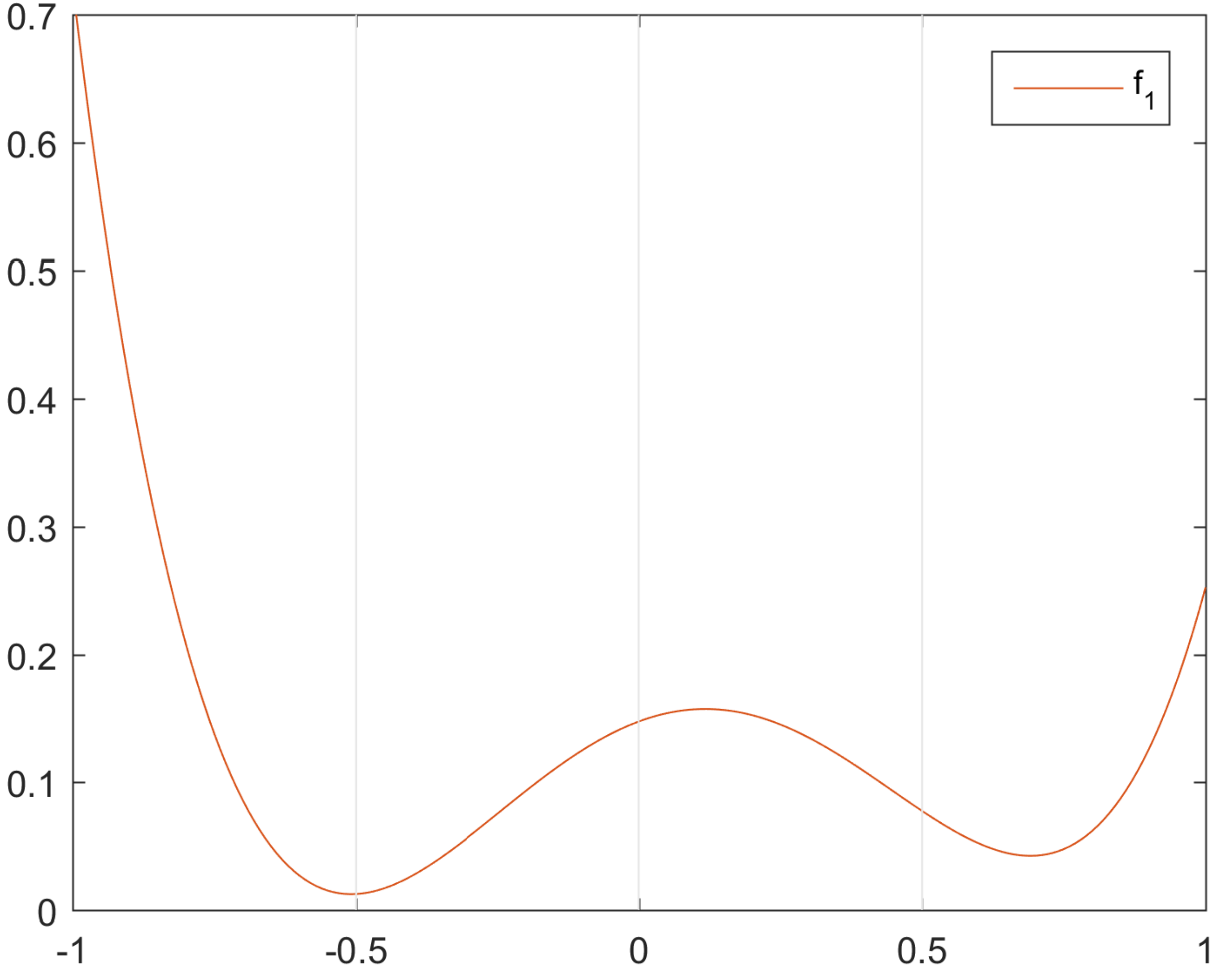

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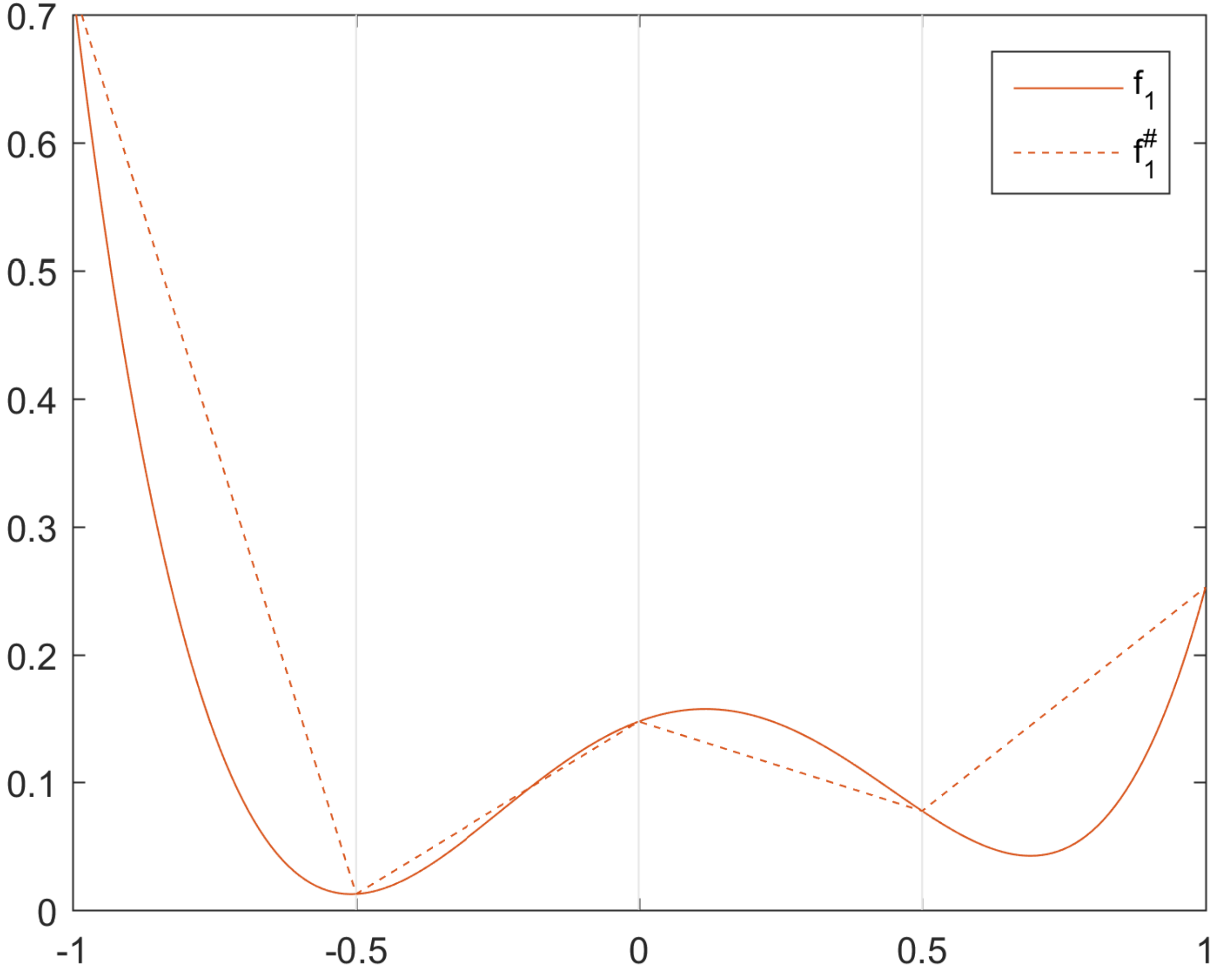

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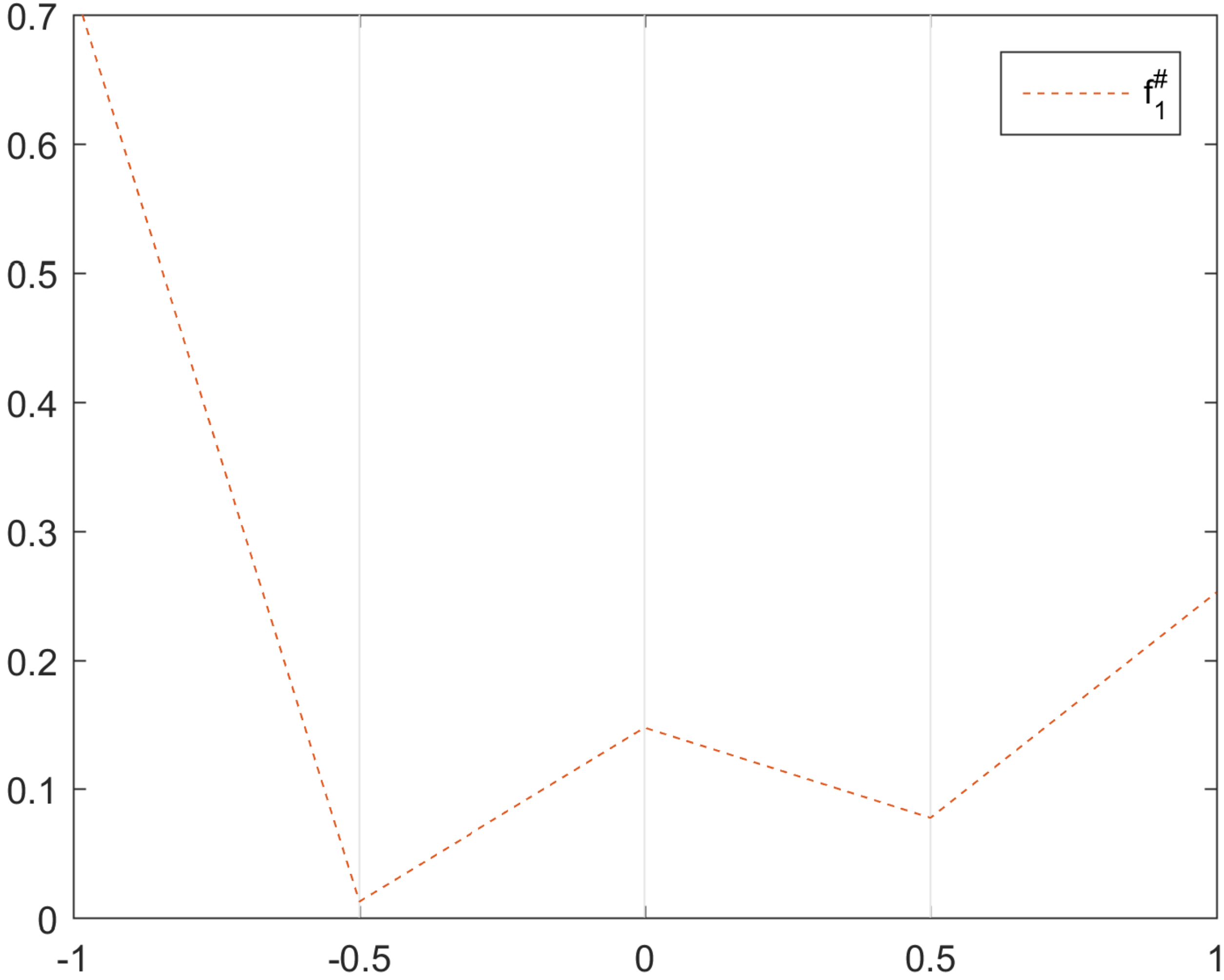

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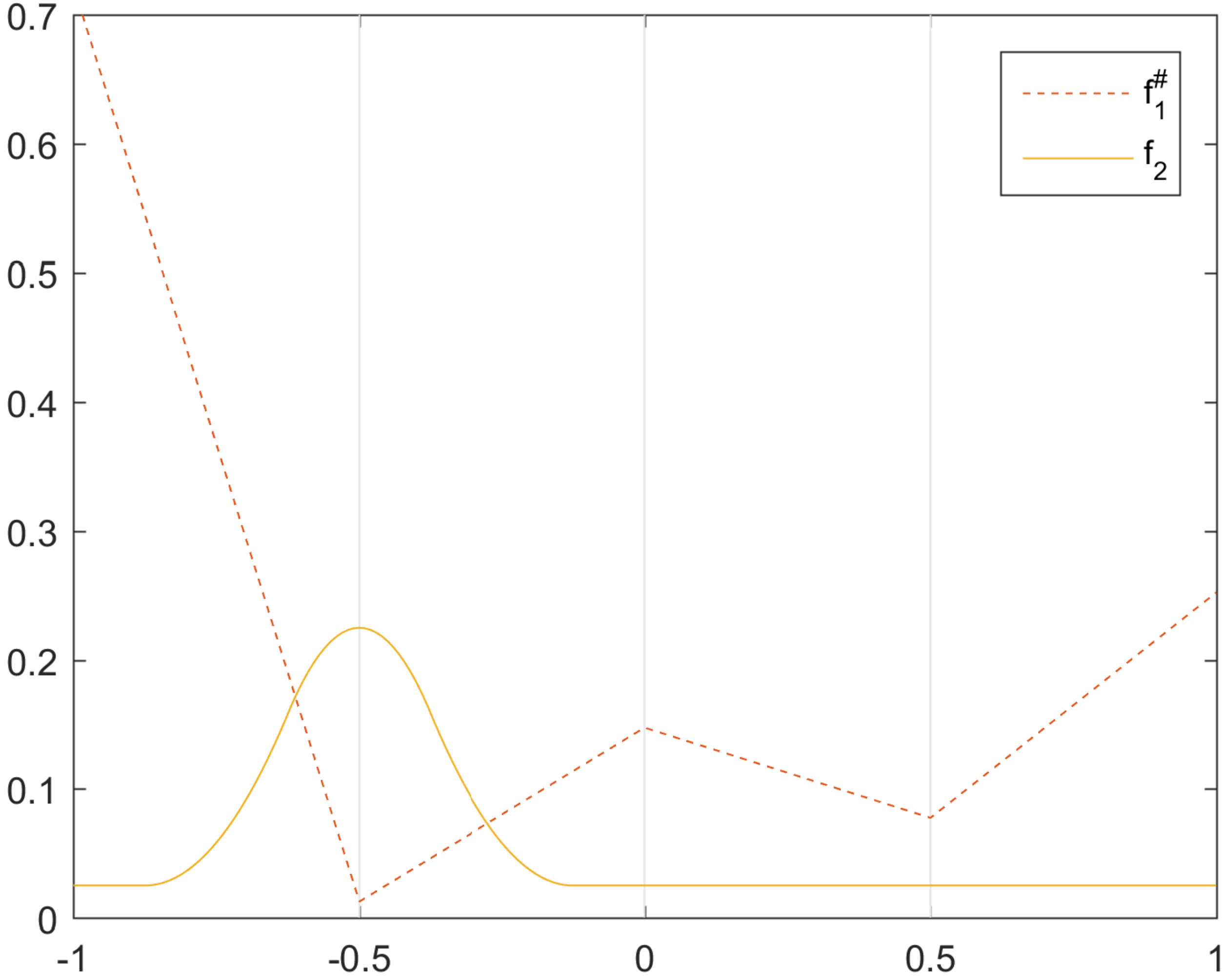

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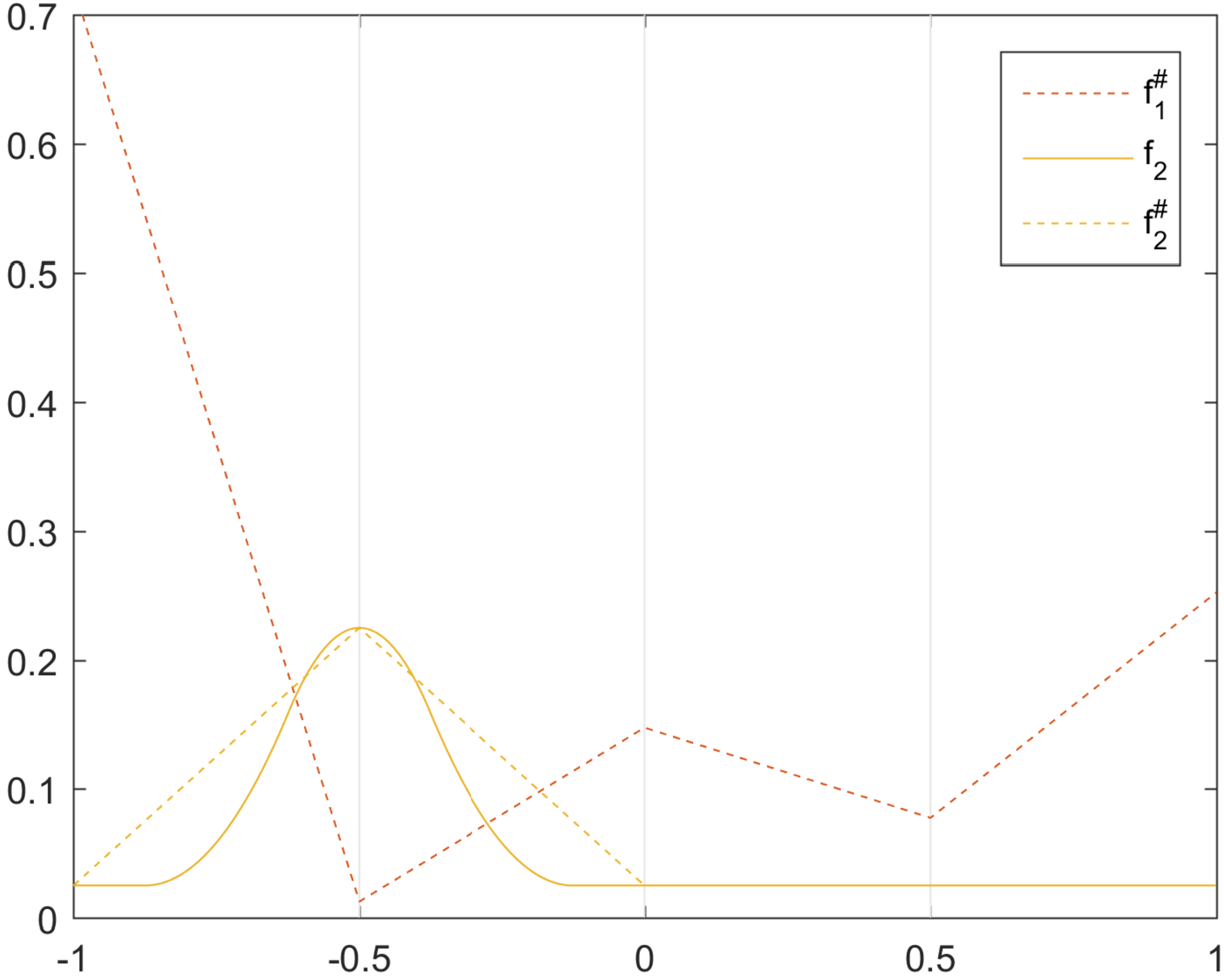

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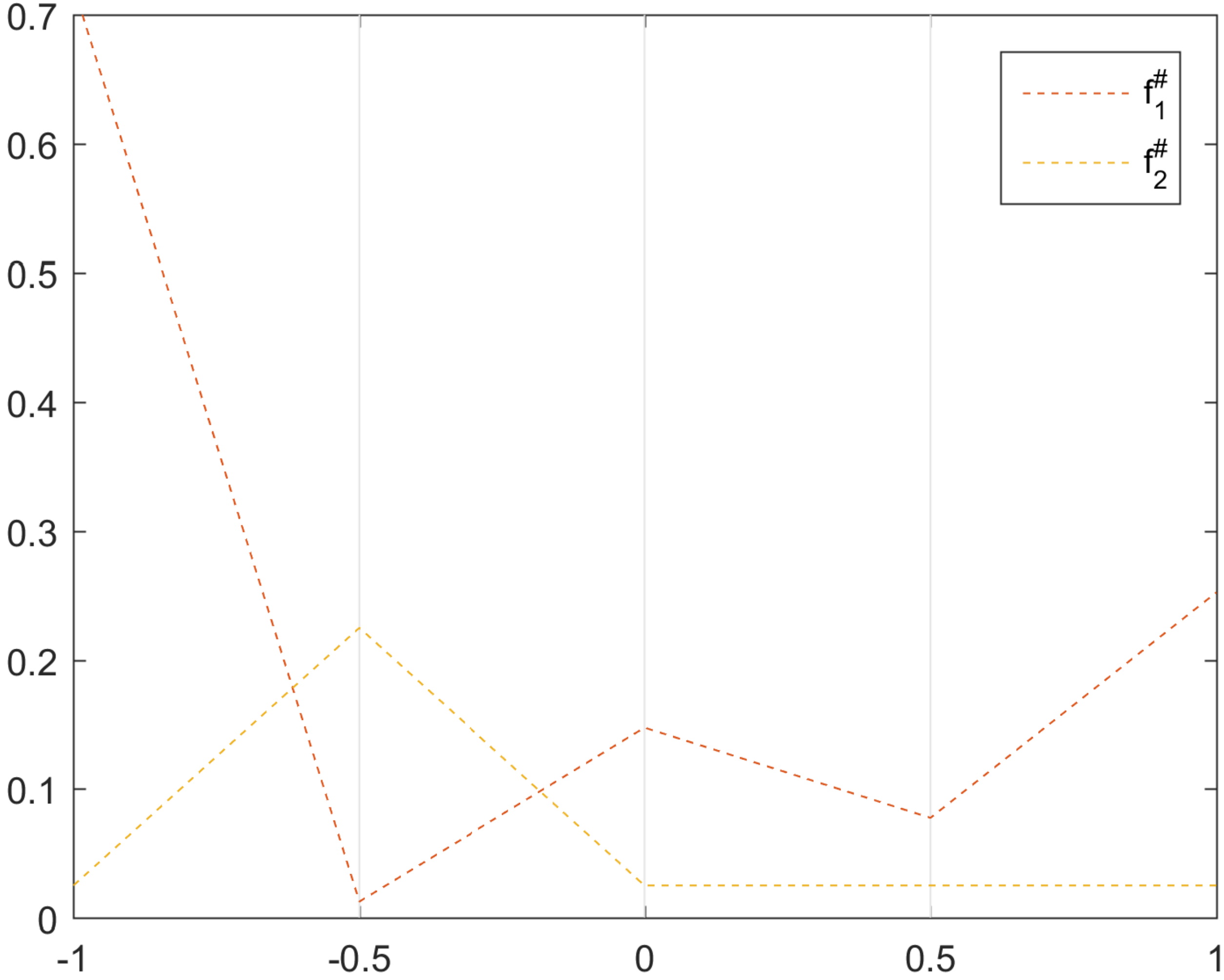

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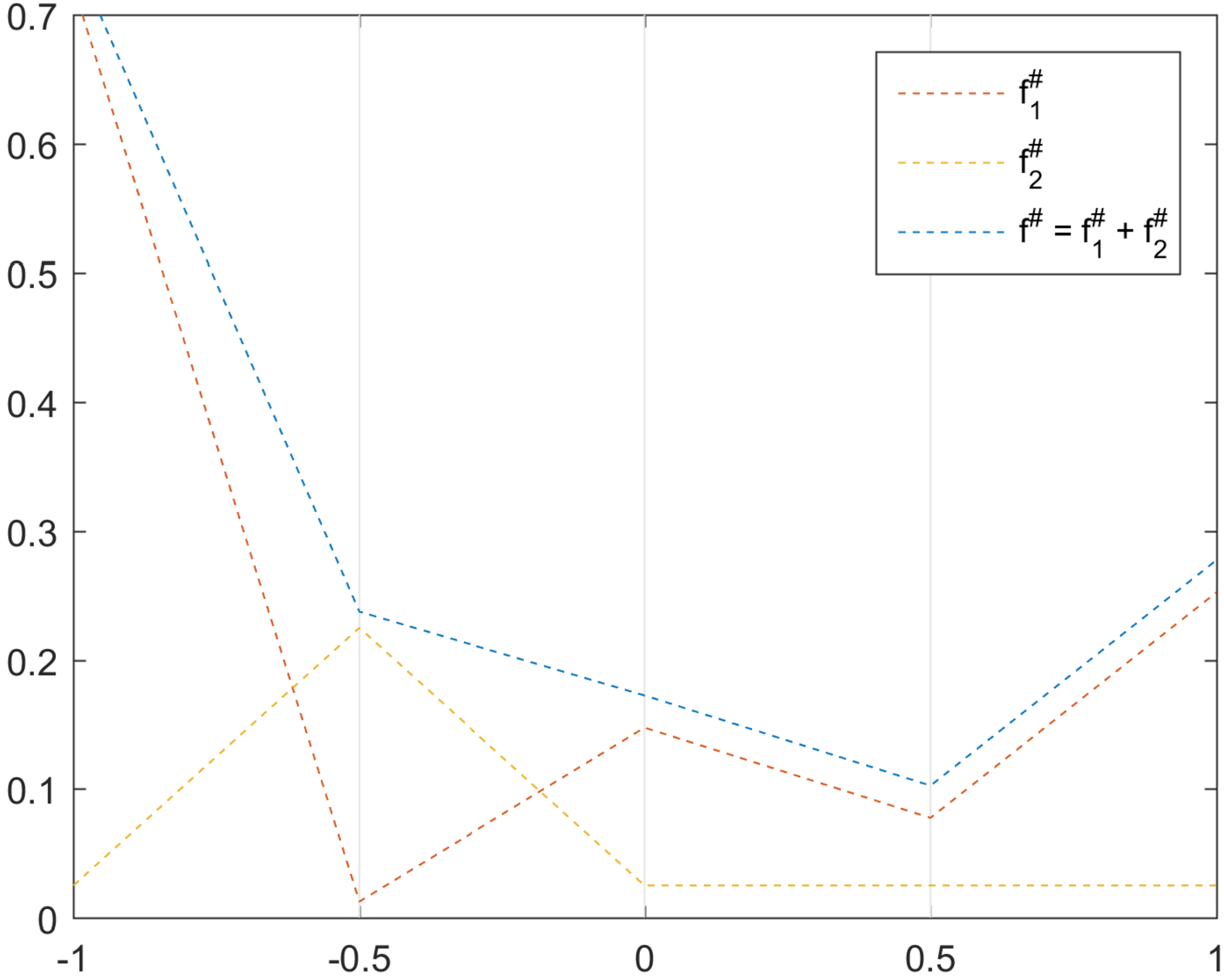

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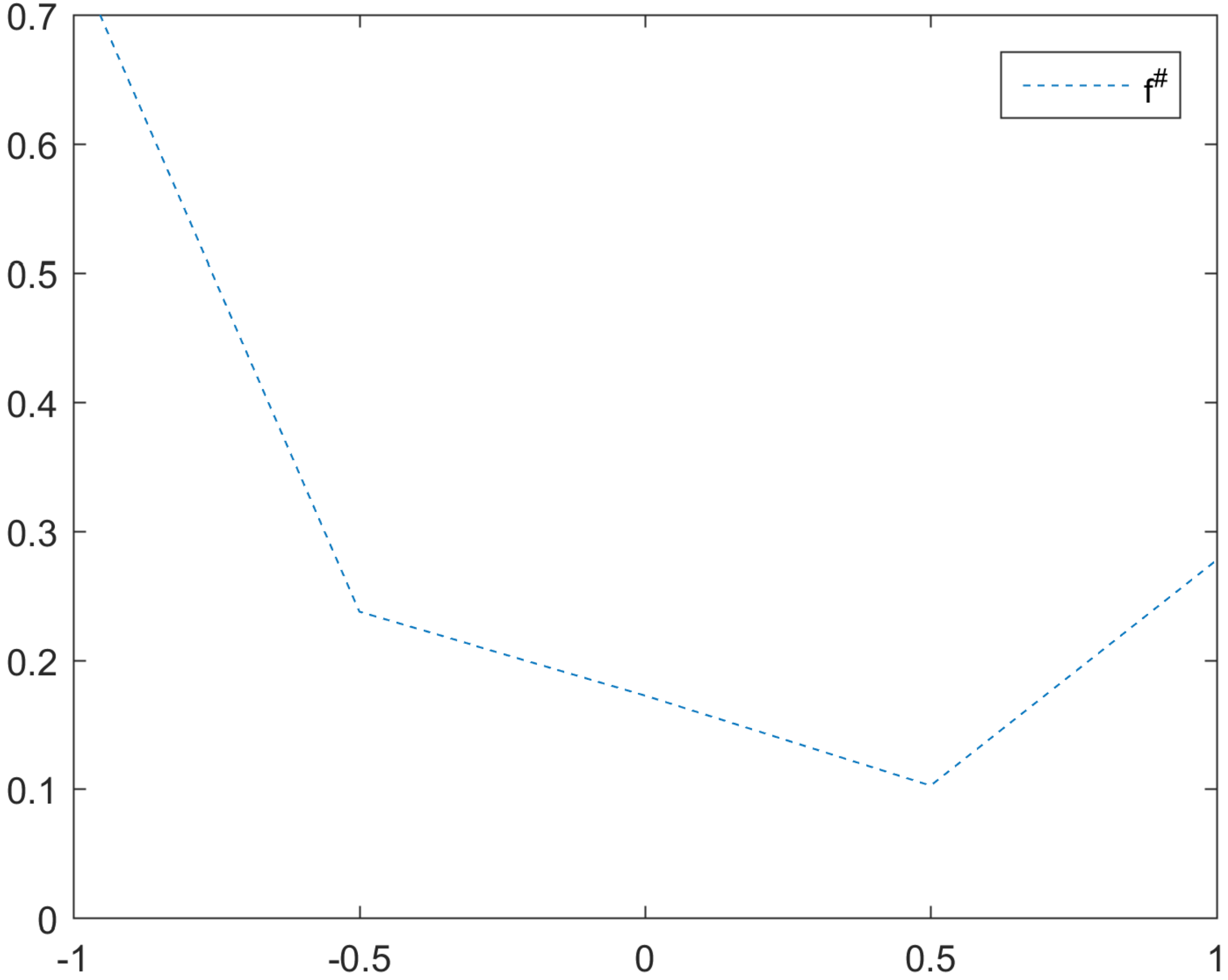

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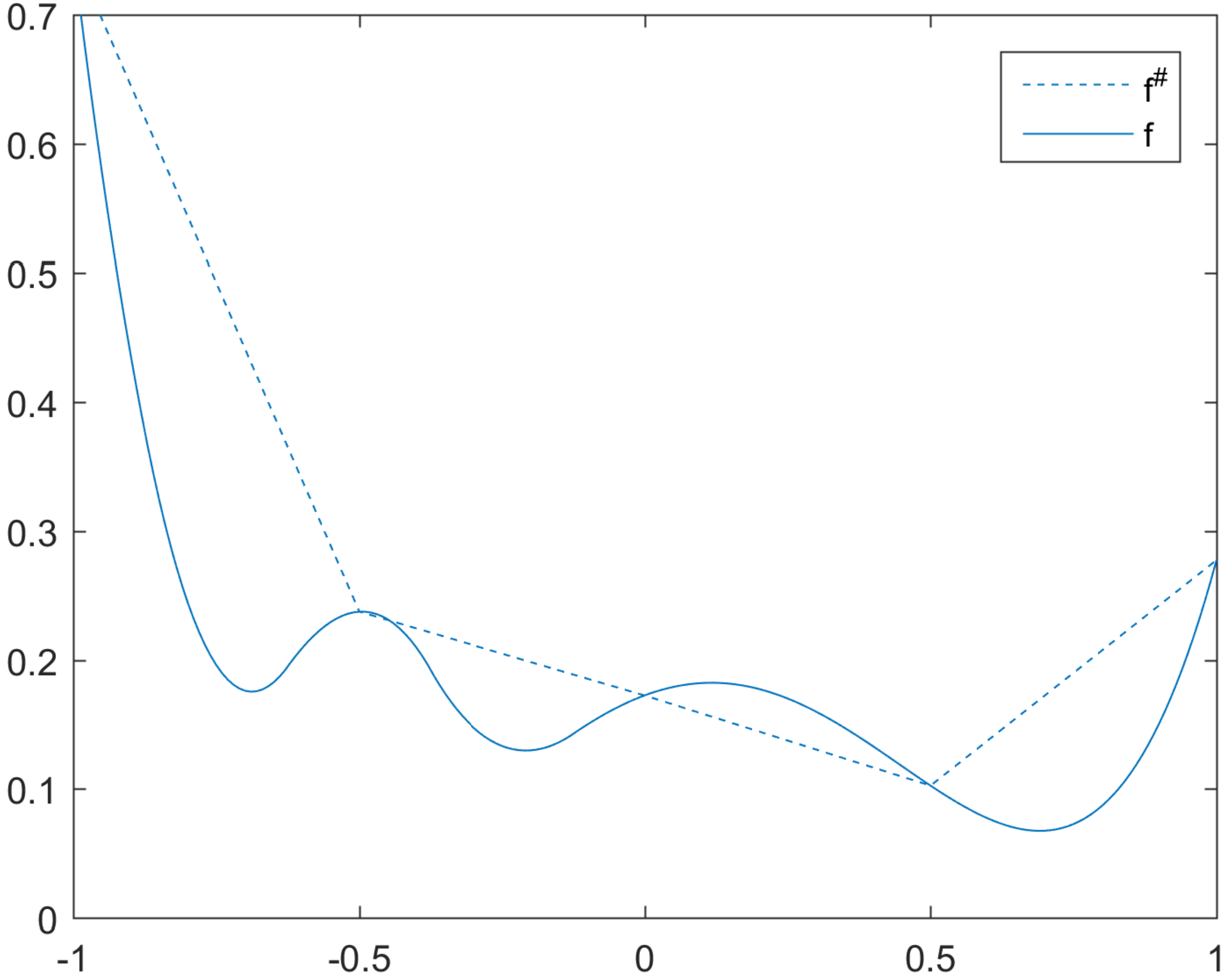

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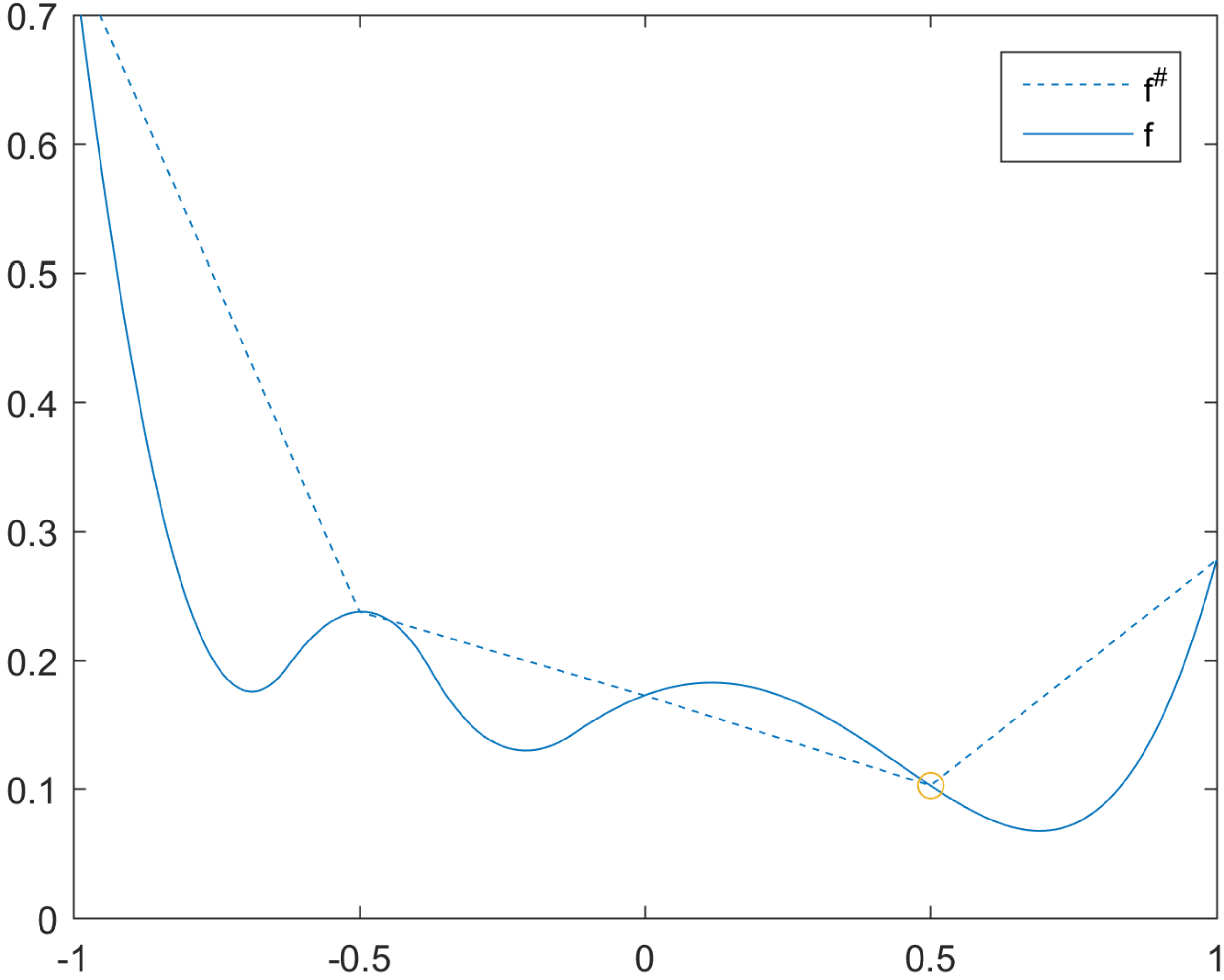
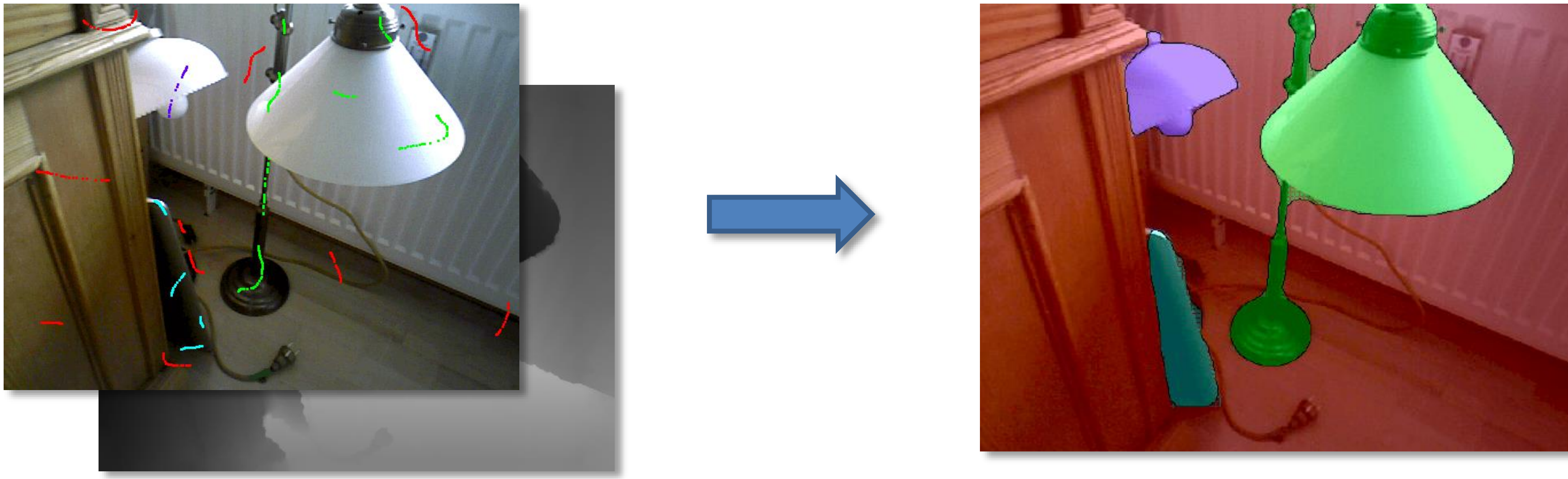
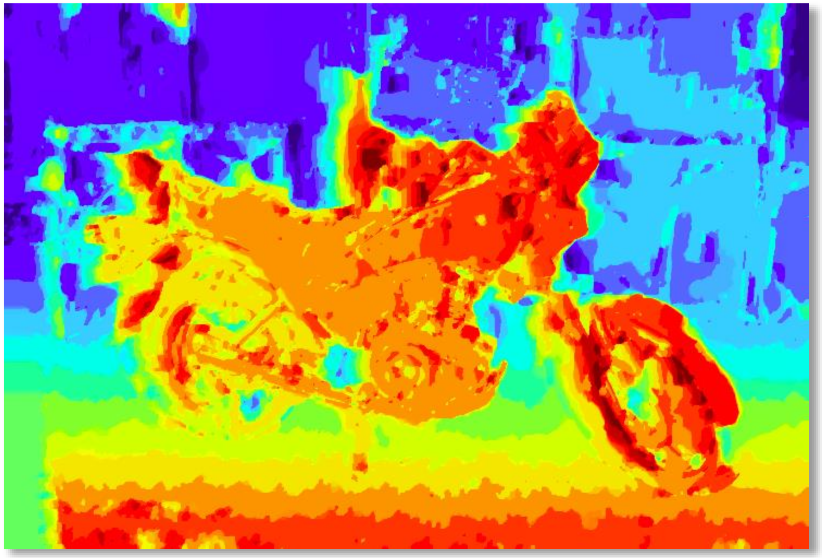




RGB-Depth Segmentation (Diebold et al. , SSVM '15)

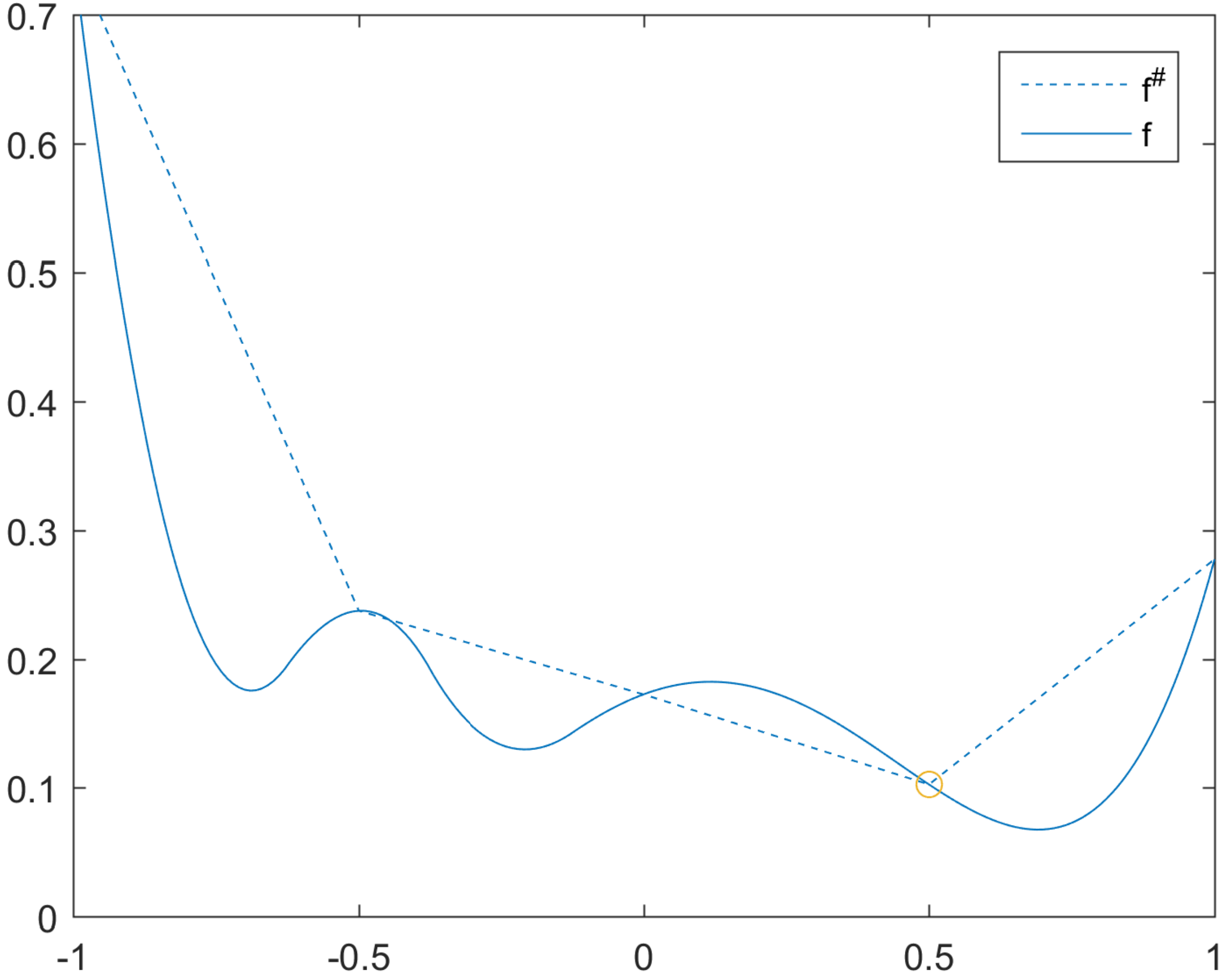
Kolev et al., Int. J. Comp. Vis. '09

Label bias

The solution tends strongly towards the chosen labels!

Lifting + linear relaxation



Precise relaxation

- Lifting + relaxation using the biconjugate:

$$\int_\Omega \rho(u(x))dx \rightsquigarrow \int_\Omega \boldsymbol{\rho}^{**}(u(x))dx$$

Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16
related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

Precise relaxation

- Lifting + relaxation using the biconjugate:

$$\int_\Omega \rho(u(x))dx \rightsquigarrow \int_\Omega \boldsymbol{\rho}^{**}(u(x))dx$$

- Linear relaxation (*1-sparse* solutions):

$$\boldsymbol{\rho}(z) = \begin{cases} \rho(t^i), & z = e^i,\ i \in \{1, \ldots, L\}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Möllenhoff, **Laude**, **Möller**, **Lellmann**, Cremers, CVPR'16
related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

Precise relaxation

- Lifting + relaxation using the biconjugate:

$$\int_\Omega \rho(u(x))dx \rightsquigarrow \int_\Omega \boldsymbol{\rho}^{**}(u(x))dx$$

- New: Precise relaxation (*2-sparse* solutions!)

$$\boldsymbol{\rho}(z) = \begin{cases} \rho((1-\alpha)t^i + \alpha t^{i+1}), & z = (1-\alpha)e^i + \alpha e^{i+1}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Möllenhoff, **Laude**, **Möller**, **Lellmann**, Cremers, CVPR'16
related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

Precise relaxation

- Lifting + relaxation using the biconjugate:

$$\int_{\Omega} \rho(u(x))dx \rightsquigarrow \int_{\Omega} \boldsymbol{\rho}^{**}(u(x))dx$$

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$$\boldsymbol{\rho}(z) = \begin{cases} \rho((1-\alpha)t^i + \alpha t^{i+1}), & z = (1-\alpha)e^i + \alpha e^{i+1}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Möllenhoff, **Laude**, **Möller**, **Lellmann**, Cremers, CVPR'16
related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

Precise relaxation

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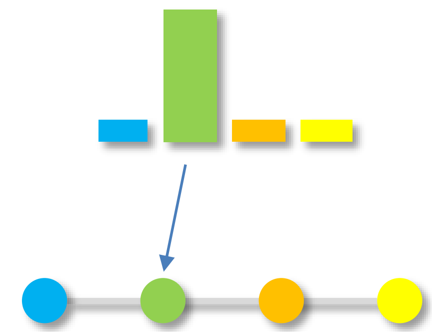

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$$\boldsymbol{\rho}(z) = \begin{cases} \rho((1-\alpha)t^i + \alpha t^{i+1}), & z = (1-\alpha)e^i + \alpha e^{i+1}, \\ +\infty, & \text{otherwise}. \end{cases}$$

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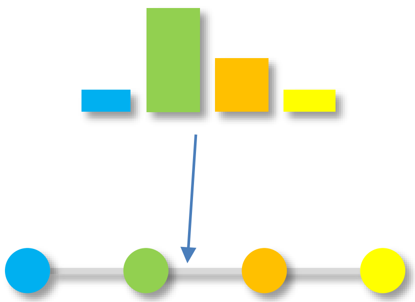

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$$\boldsymbol{\rho}(z) = \begin{cases} \rho((1-\alpha)t^i + \alpha t^{i+1}), & z = (1-\alpha)e^i + \alpha e^{i+1}, \\ +\infty, & \text{otherwise.} \end{cases}$$

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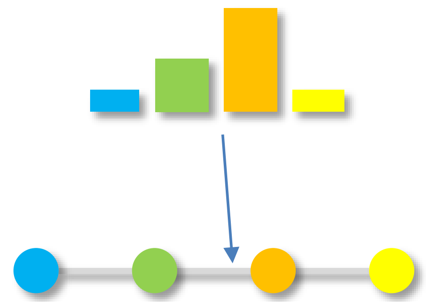

- New: Precise relaxation (*2-sparse* solutions!)

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Precise relaxation

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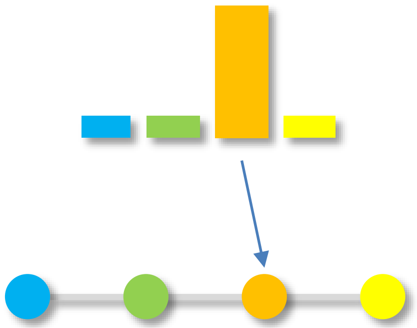

- New: Precise relaxation (*2-sparse* solutions!)

$$\boldsymbol{\rho}(z) = \begin{cases} \rho((1-\alpha)t^i + \alpha t^{i+1}), & z = (1-\alpha)e^i + \alpha e^{i+1}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Möllenhoff, **Laude**, **Möller**, **Lellmann**, Cremers, CVPR'16
related: Zach, Kohli, ECCV'12; Zach, AISTATS'13

New: lifting + precise relaxation

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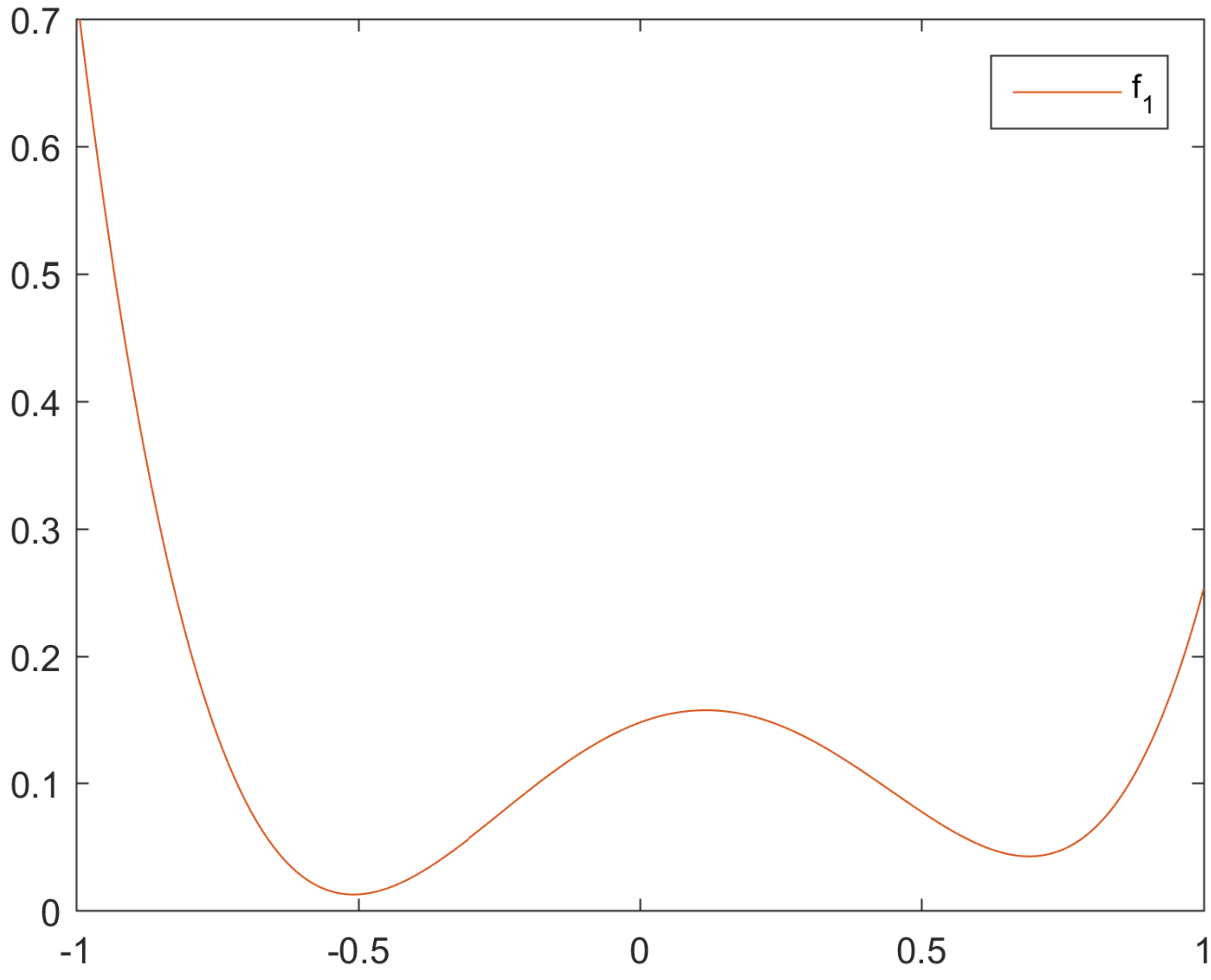
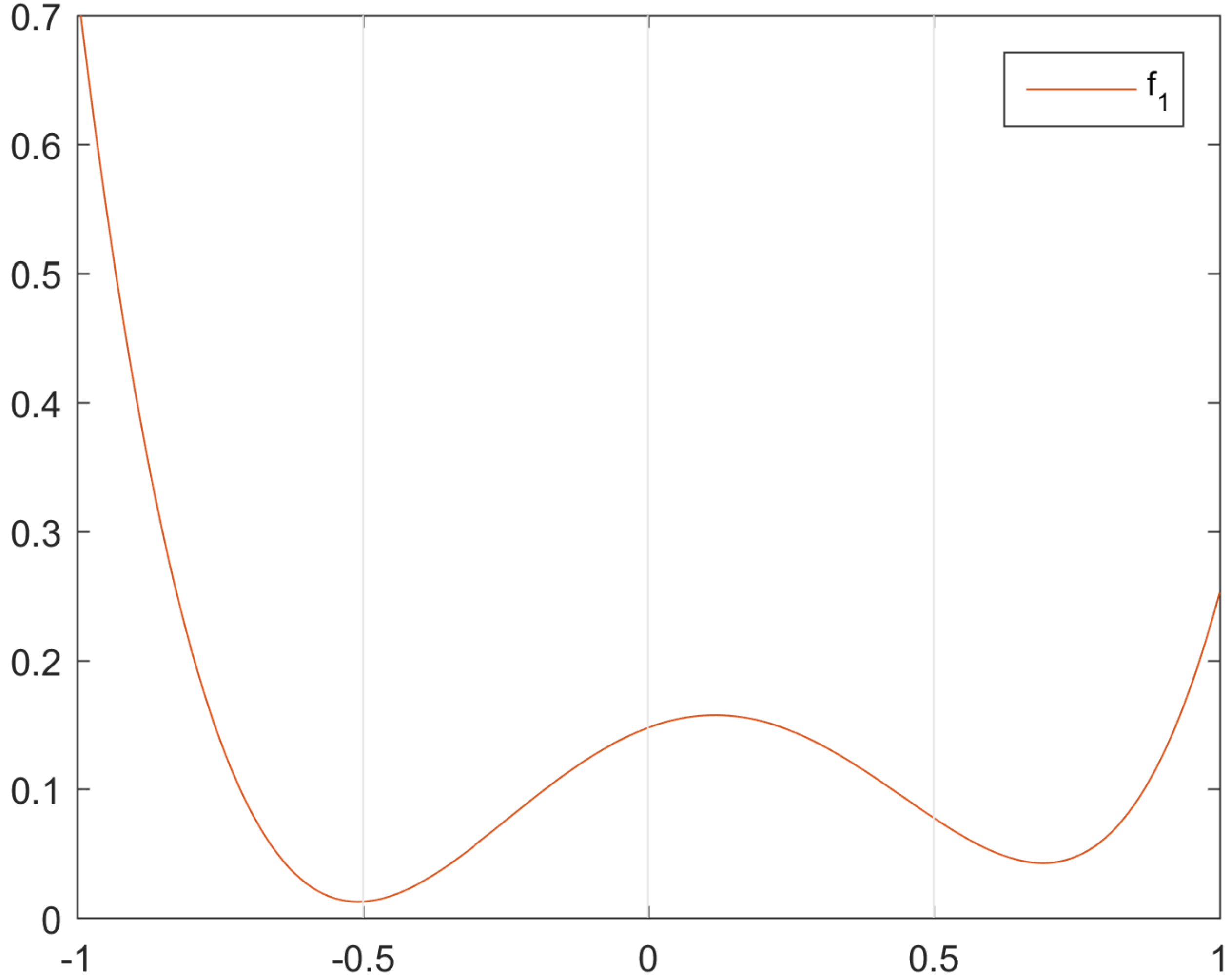
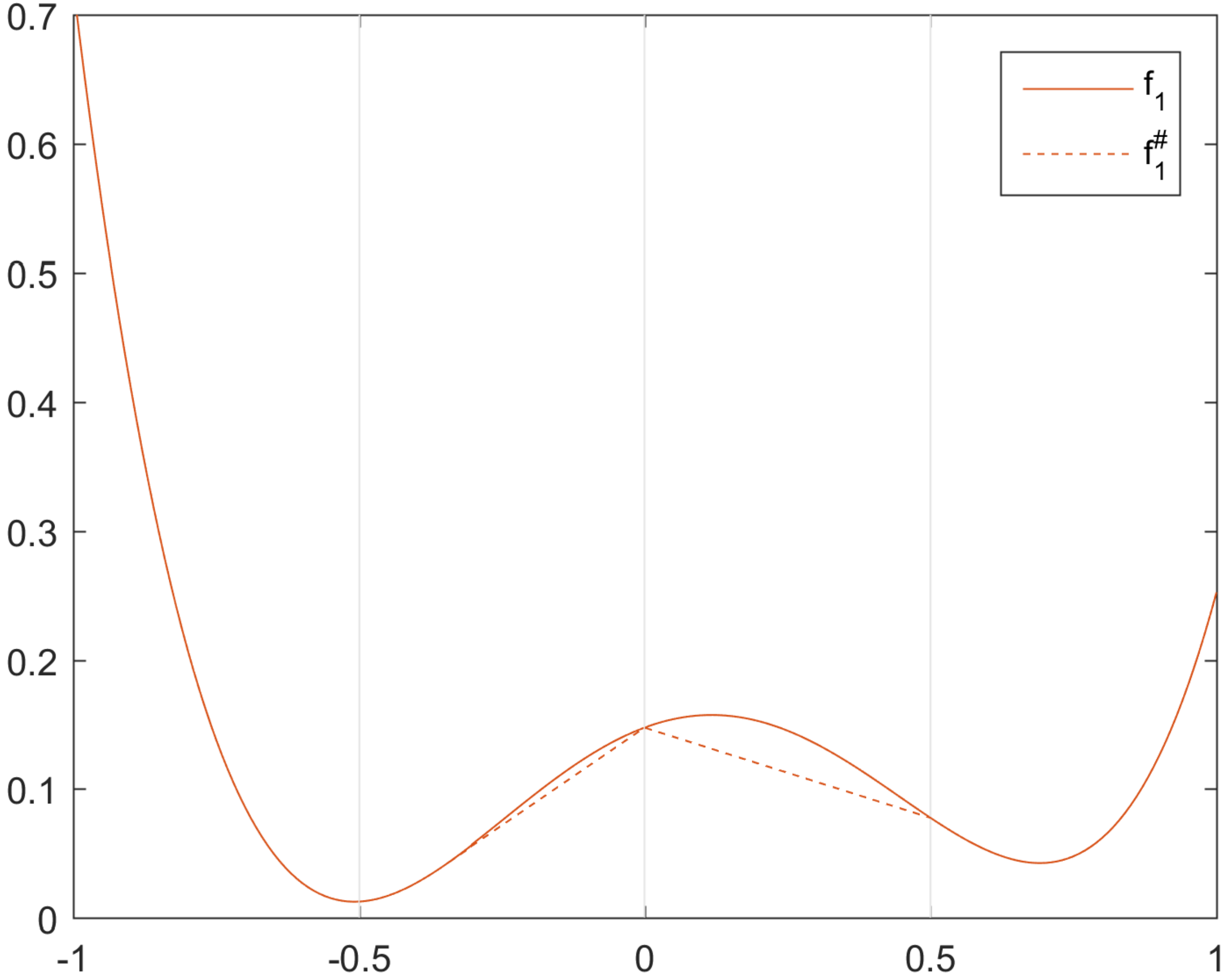


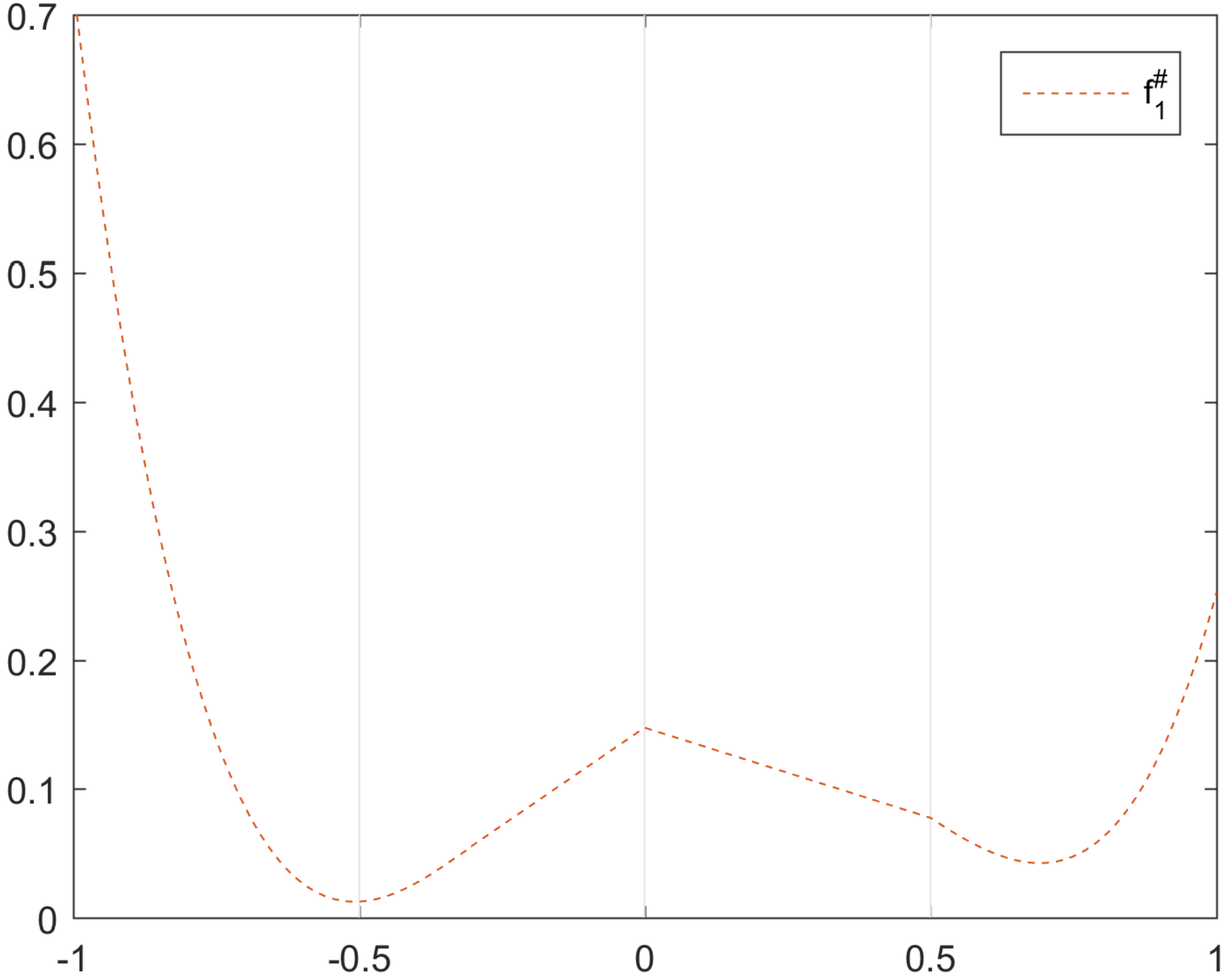
New: lifting + precise relaxation



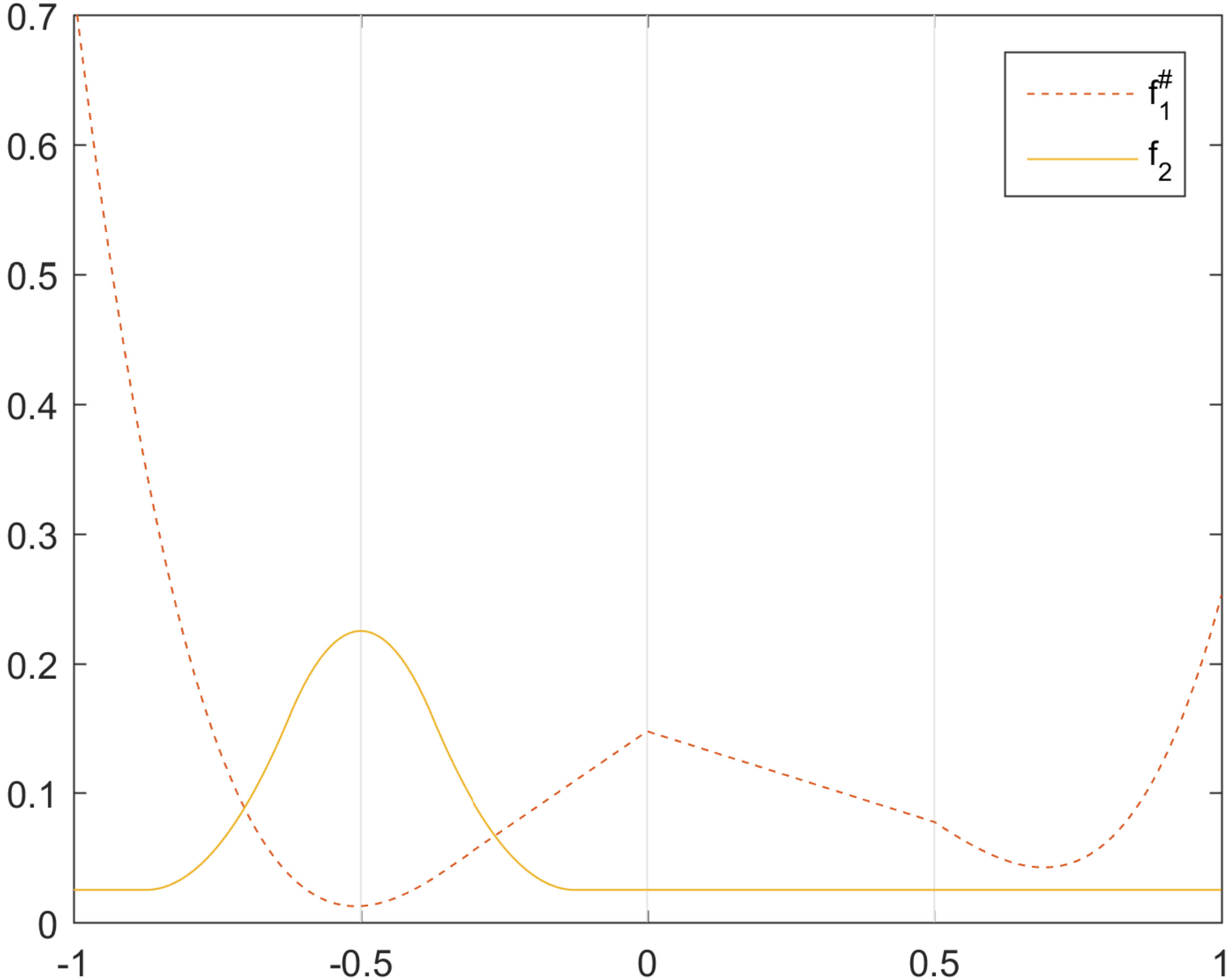
New: lifting + precise relaxation



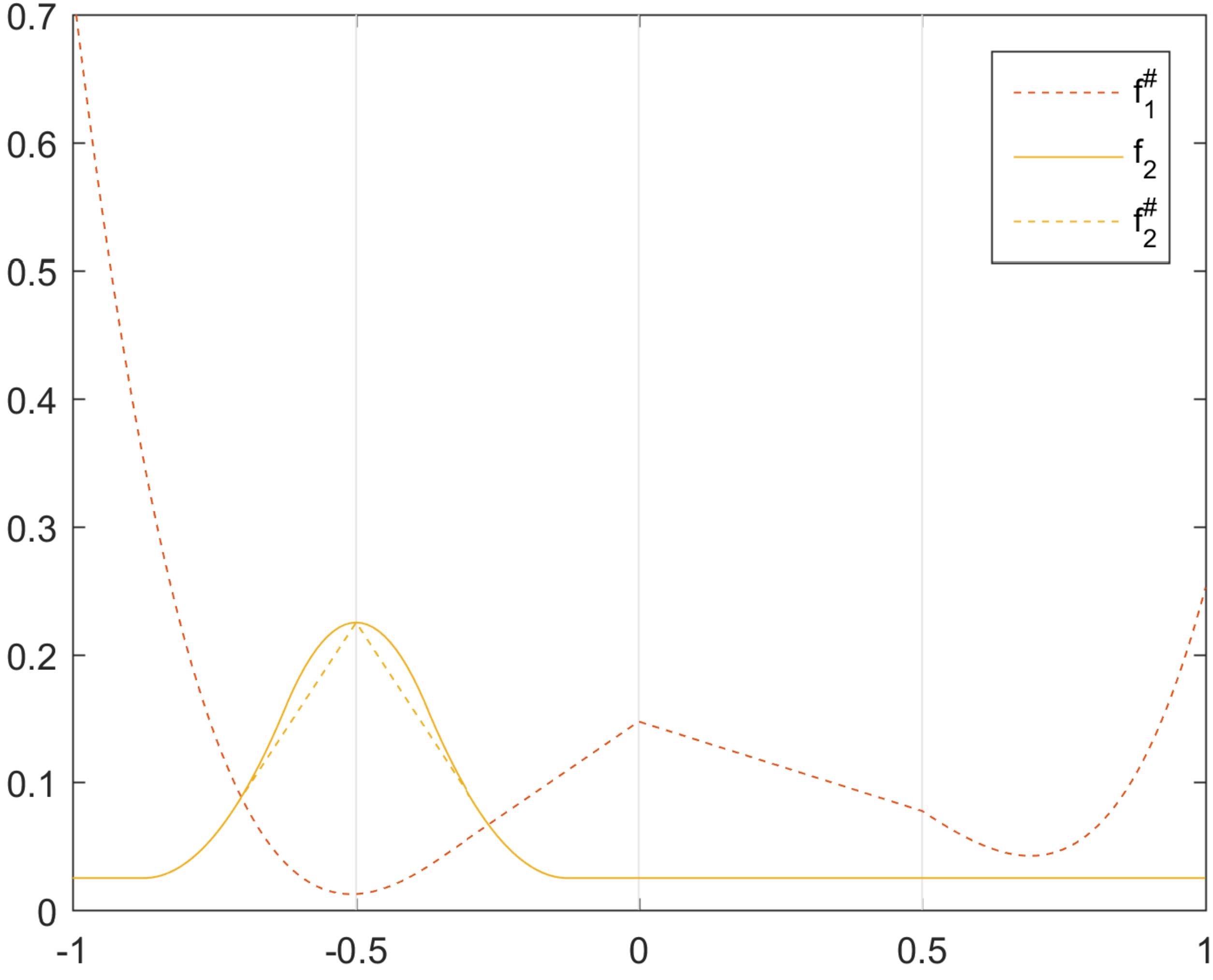
New: lifting + precise relaxation



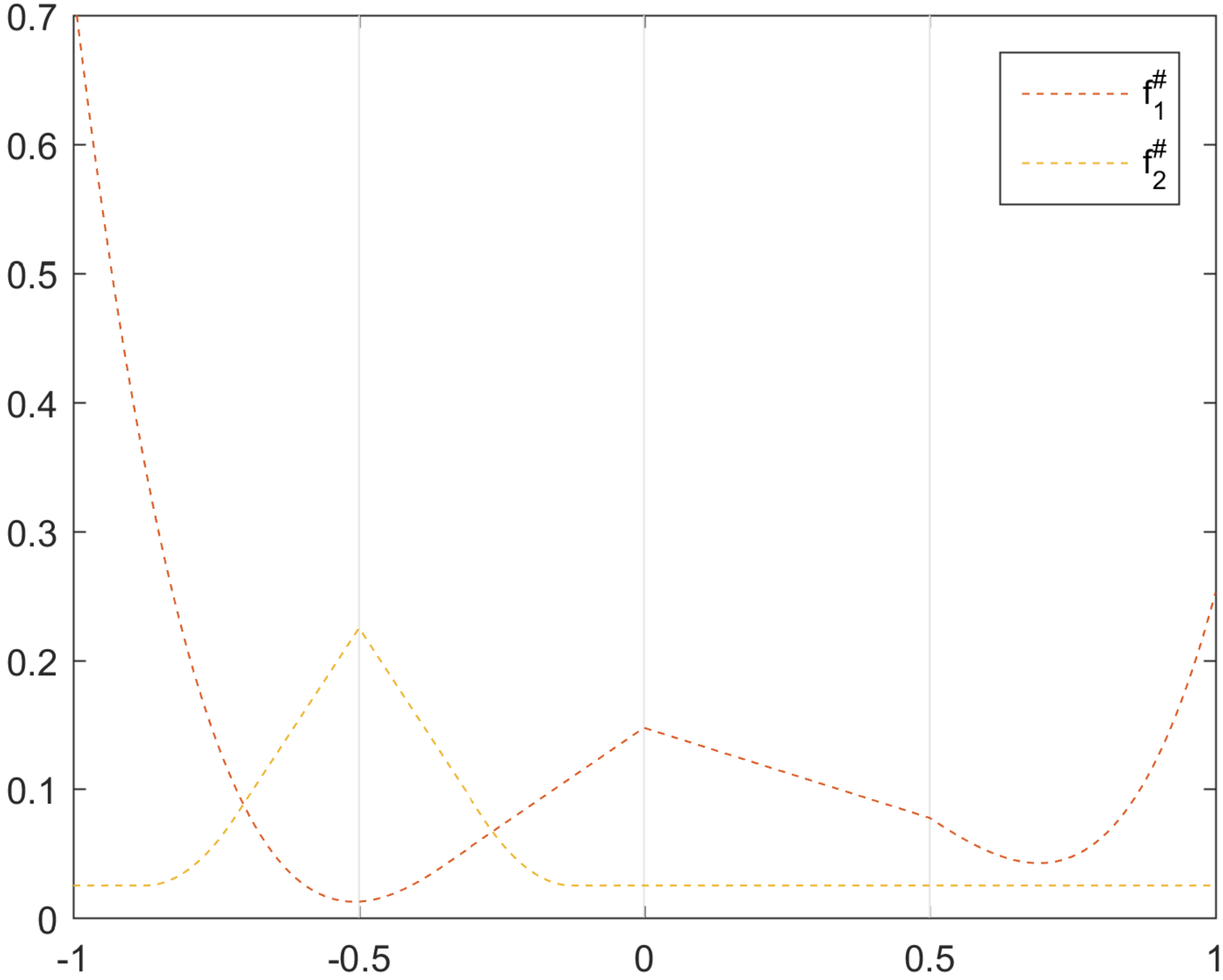
New: lifting + precise relaxation



New: lifting + precise relaxation



New: lifting + precise relaxation



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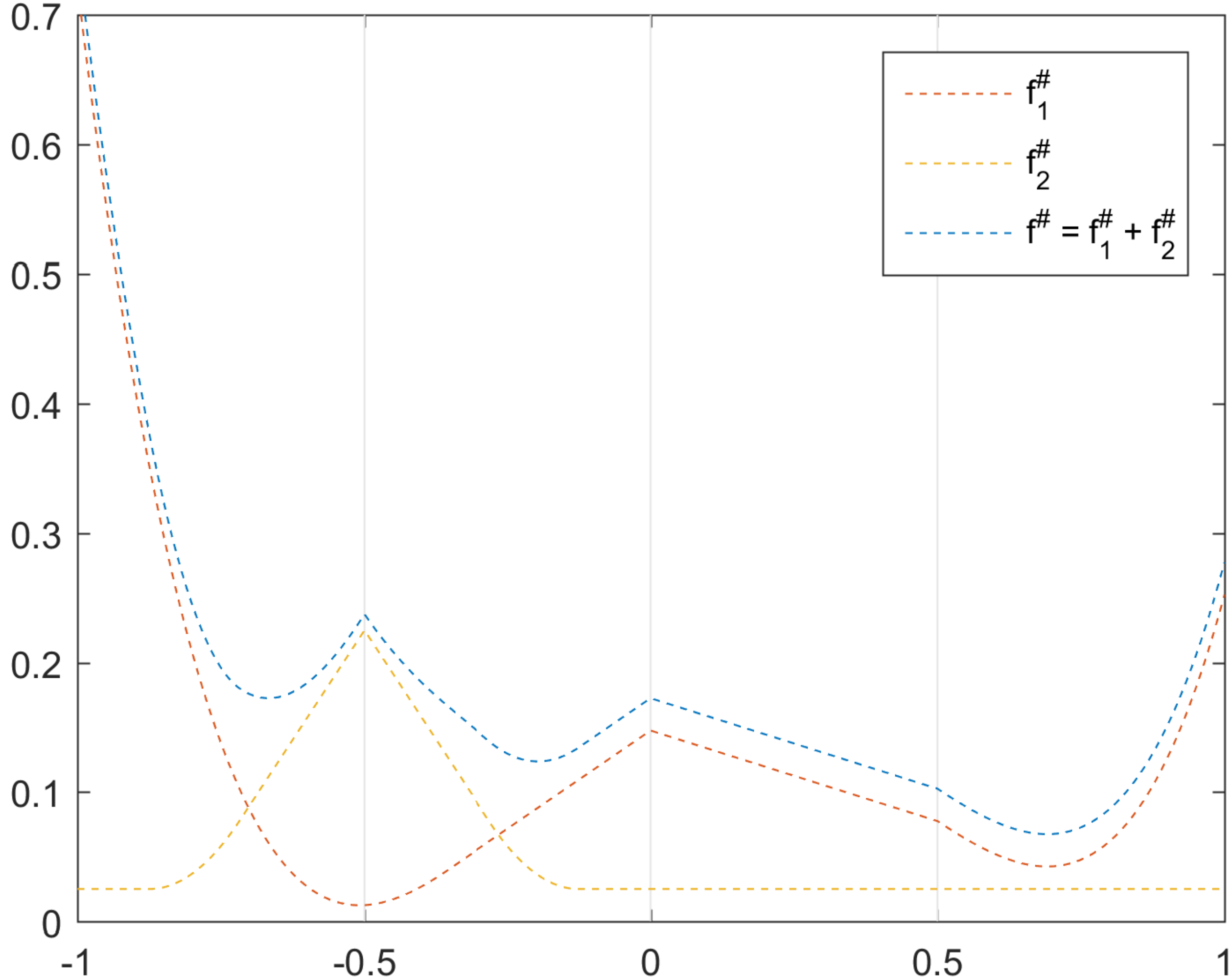
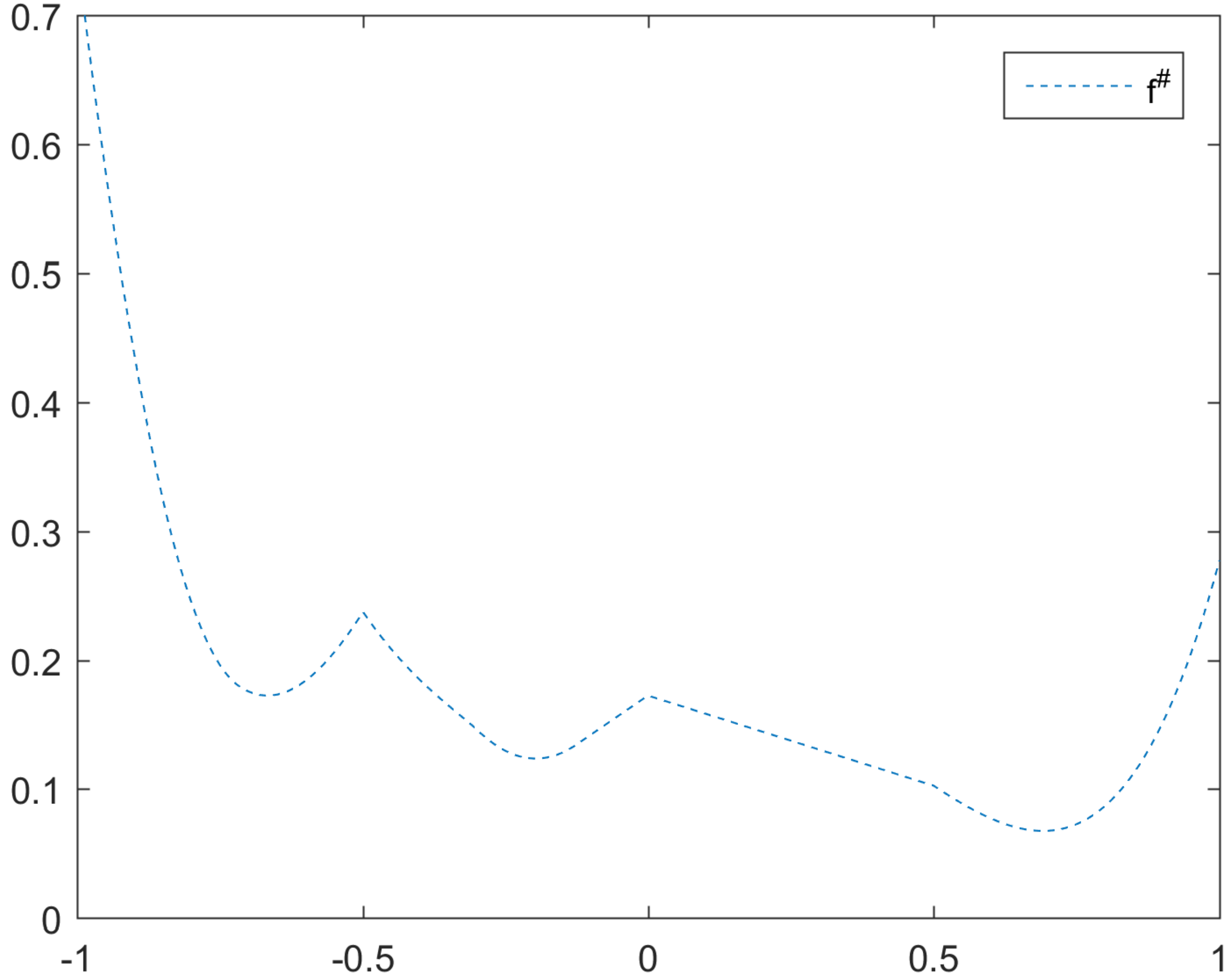
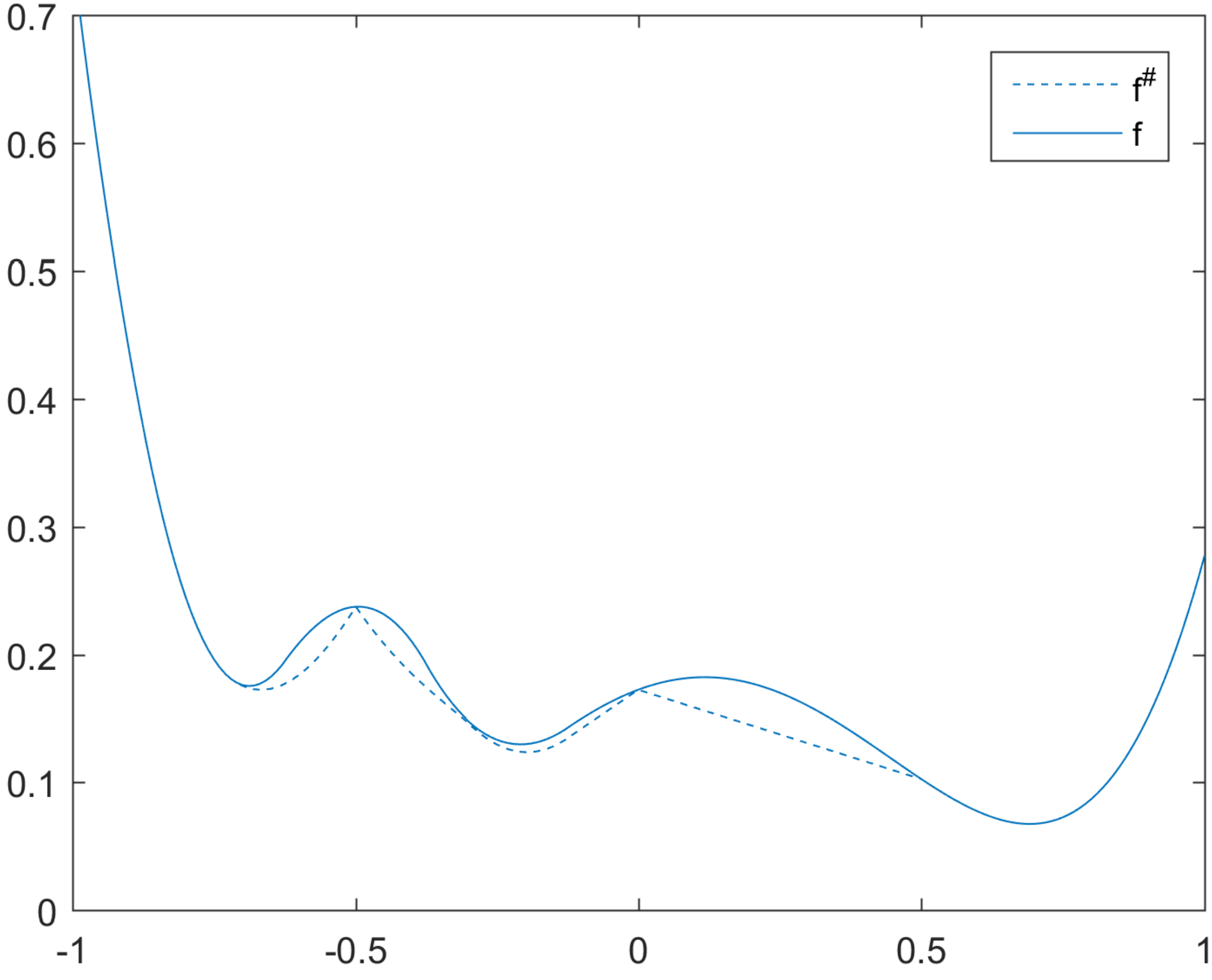


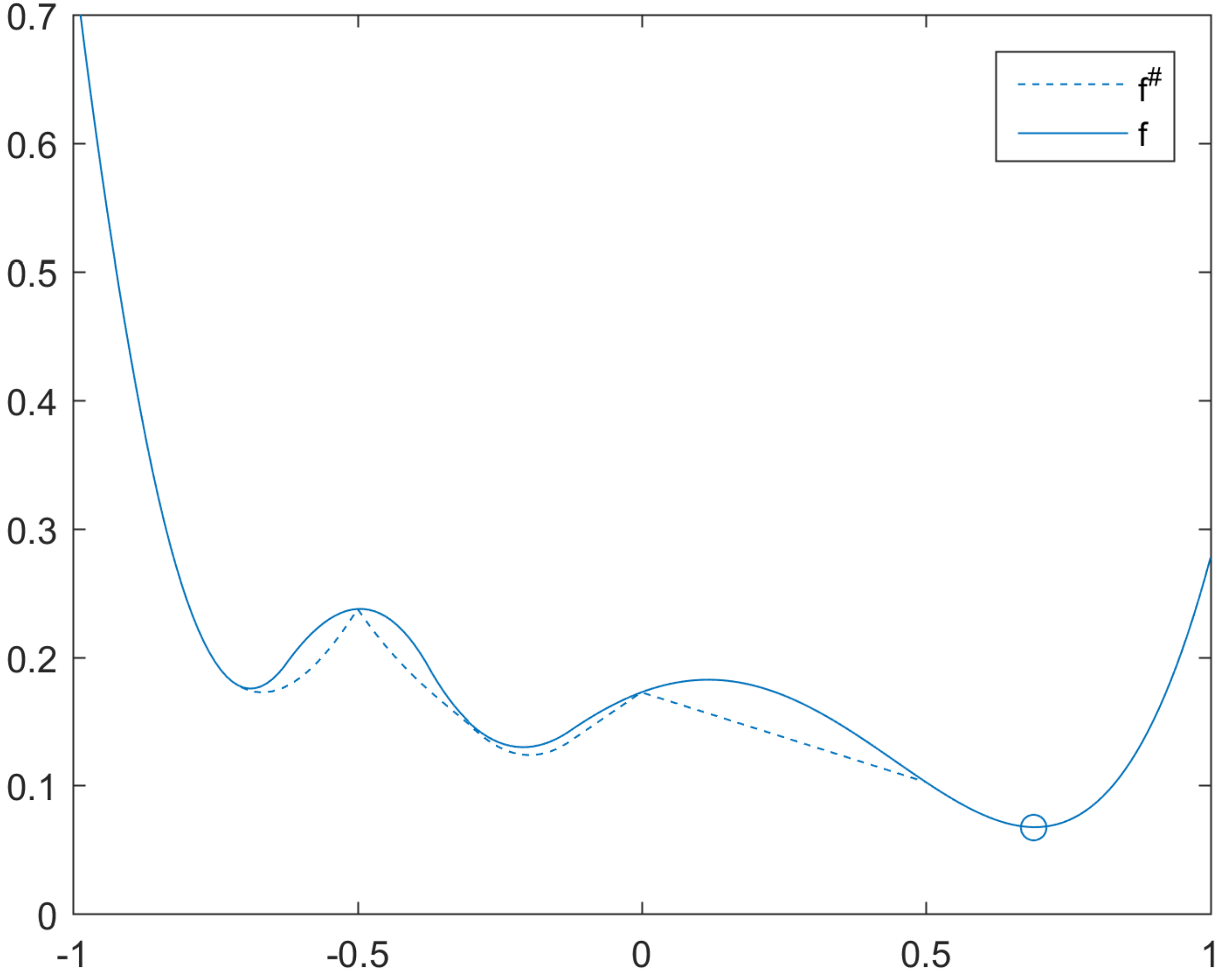
New: lifting + precise relaxation



New: lifting + precise relaxation



New: lifting + precise relaxation



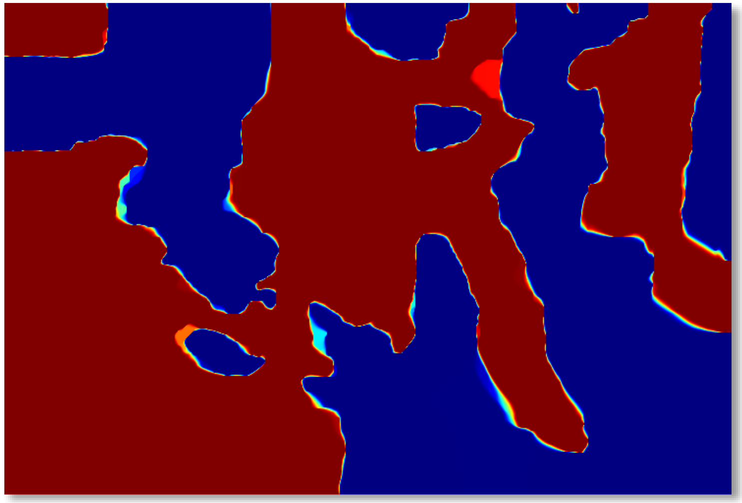
Results – Depth from stereo

Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR’16

Results – Depth from stereo

linear lifting

$L = 2$

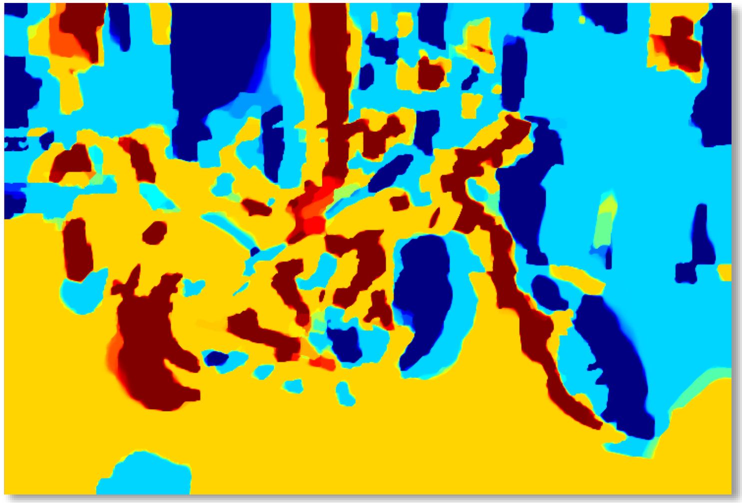


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

linear lifting

$L = 4$

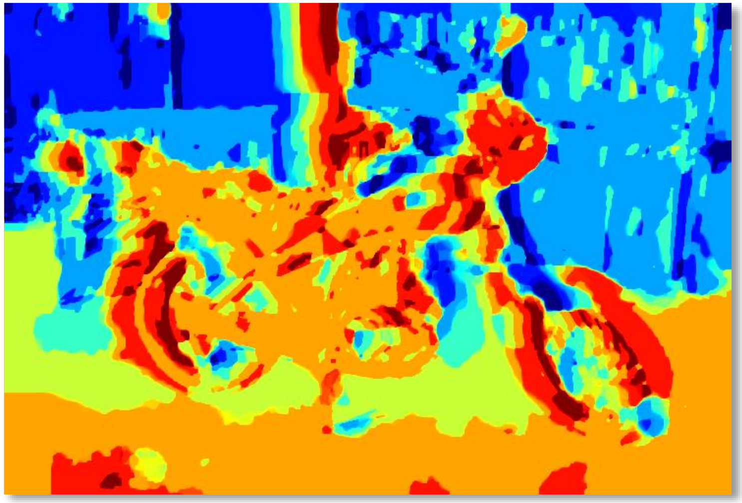


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

linear lifting

$L = 8$

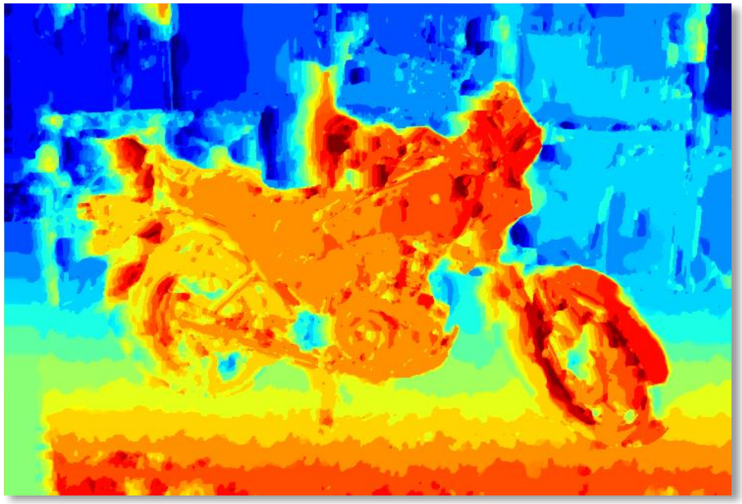


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

linear lifting

$L = 16$

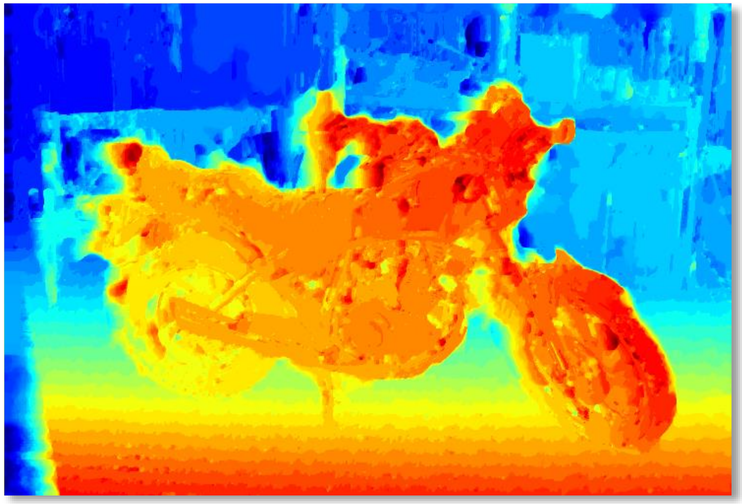


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

linear lifting

$L = 32$

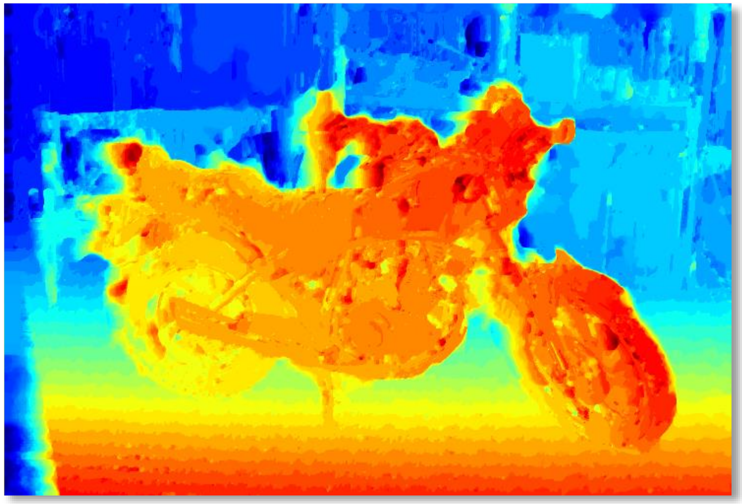


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

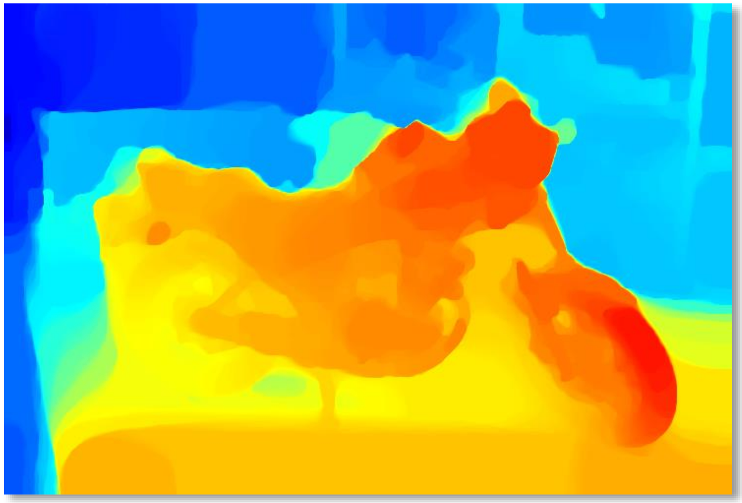
linear lifting

$L = 32$



precise lifting

$L = 2$

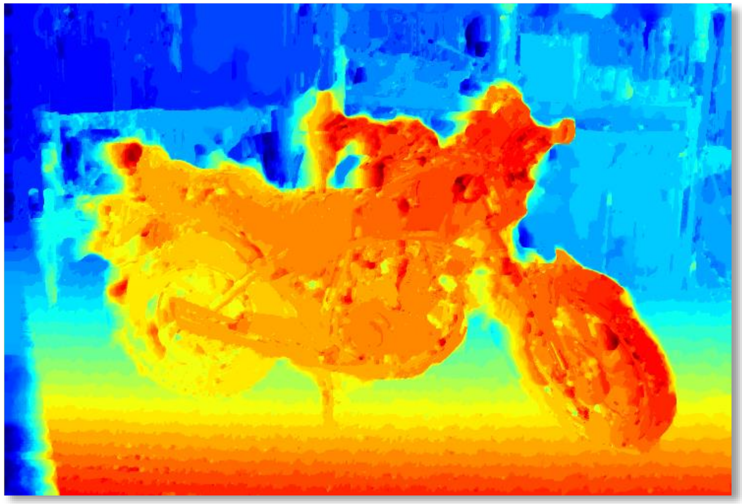


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

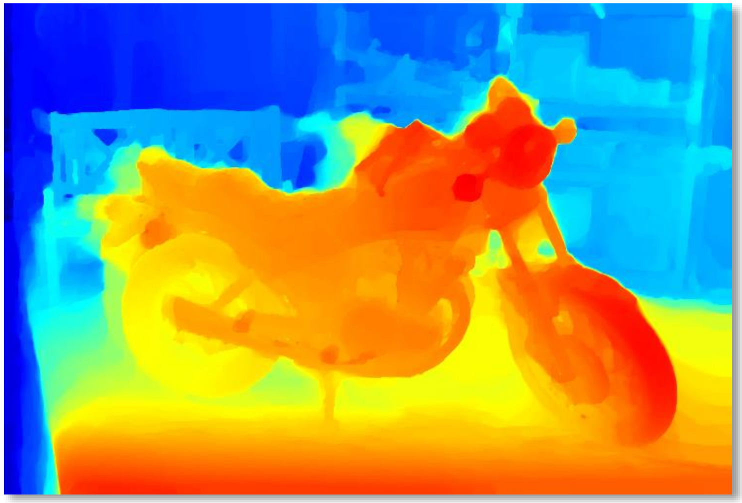
linear lifting

$L = 32$



precise lifting

$L = 4$

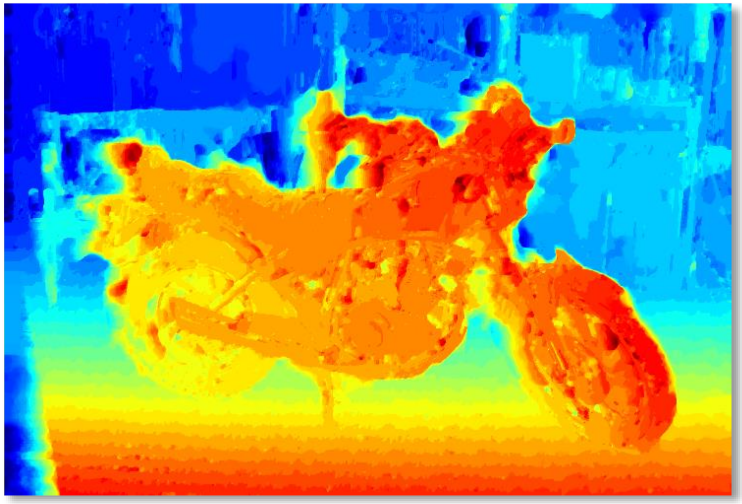


Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Results – Depth from stereo

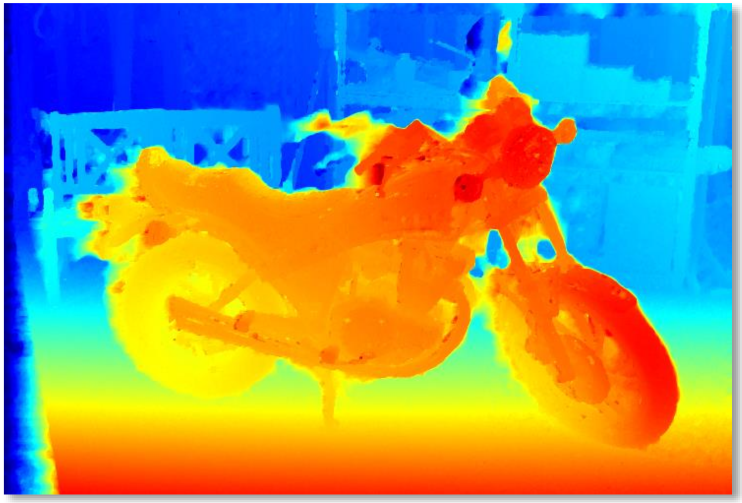
linear lifting

$L = 32$



precise lifting

$L = 8$



Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

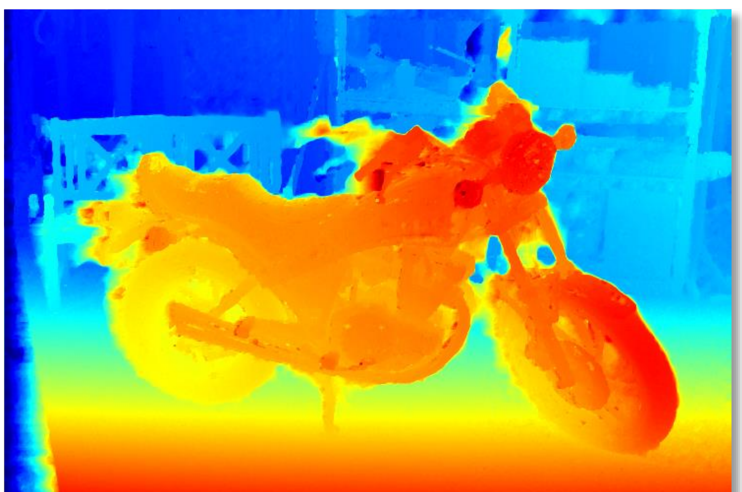
Results – Depth from stereo

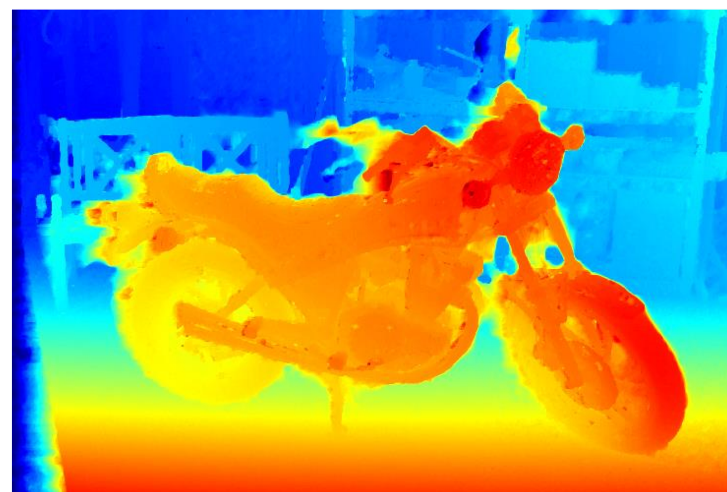
linear lifting

$L = 270$

precise lifting

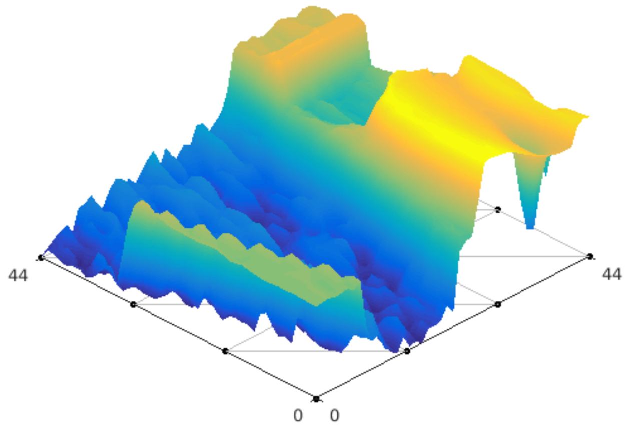
$L = 8$



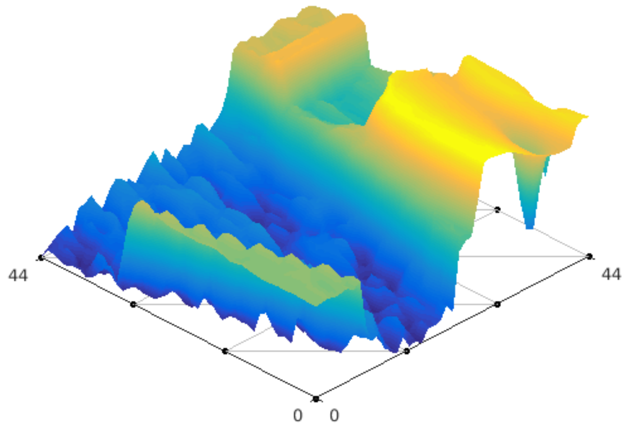


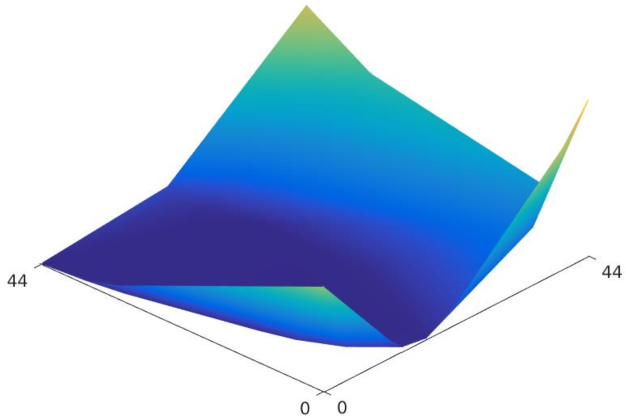
Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR'16

Multiple dimensions

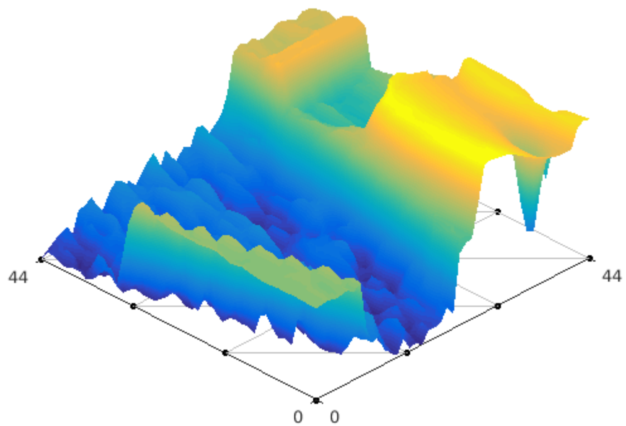


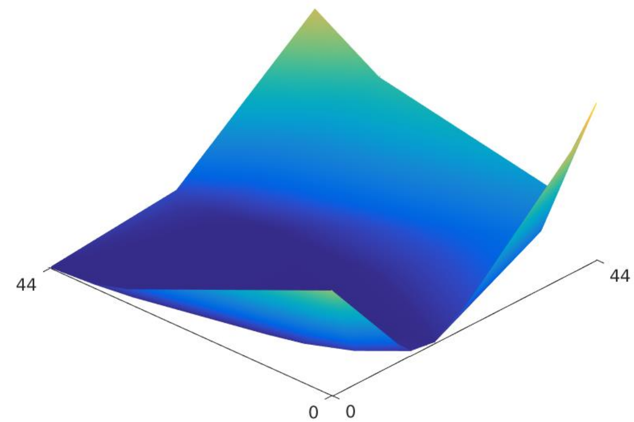
Multiple dimensions

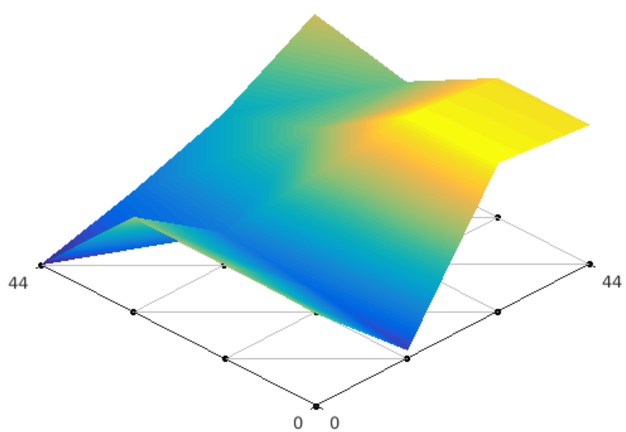




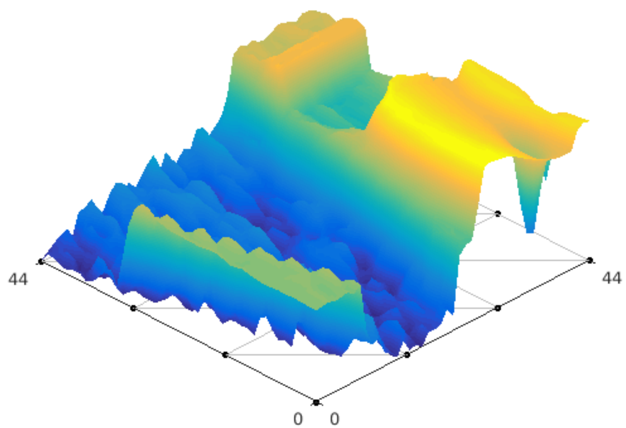
Multiple dimensions

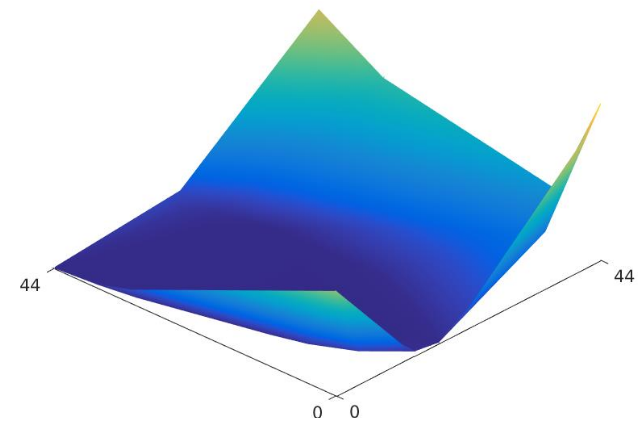


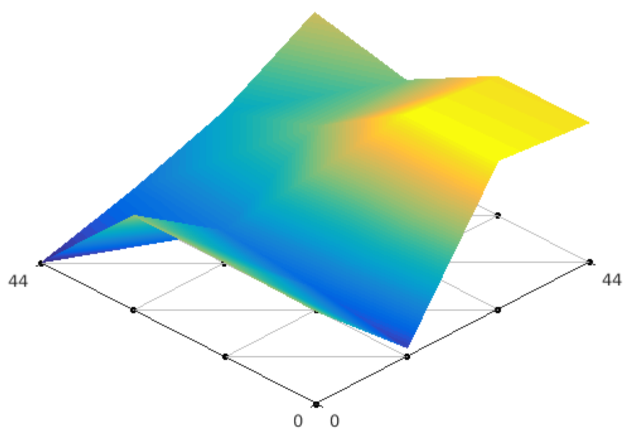


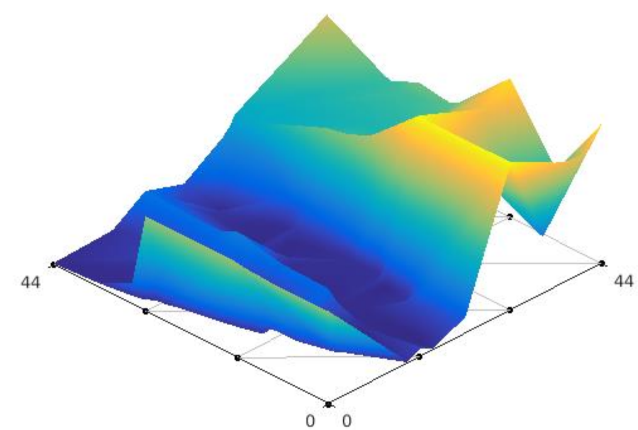


Multiple dimensions









Color denoising

input

Color denoising

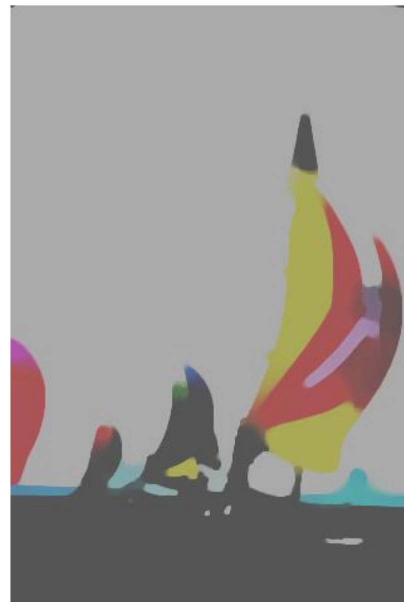
input

exact solution

Color denoising

input

exact solution

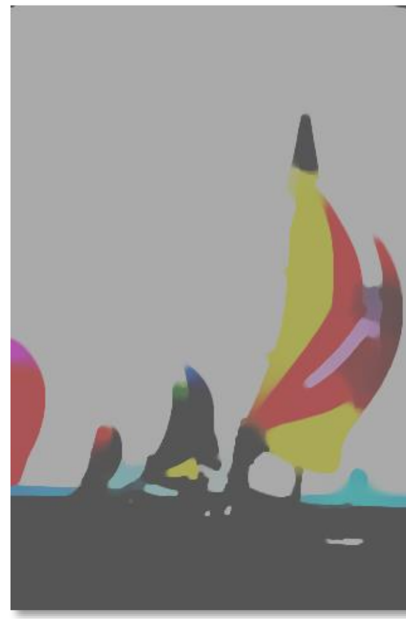


lifted linear
64 labels

Color denoising

input

exact solution

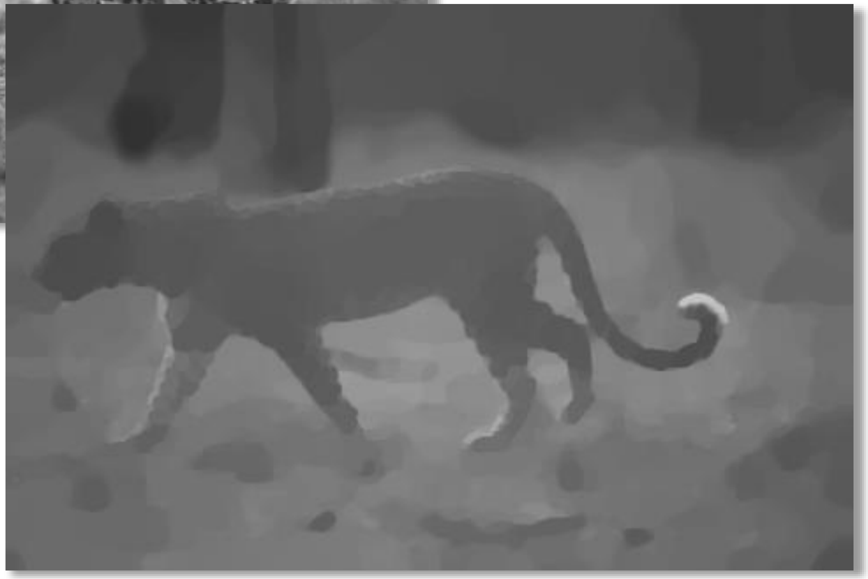


lifted linear
64 labels

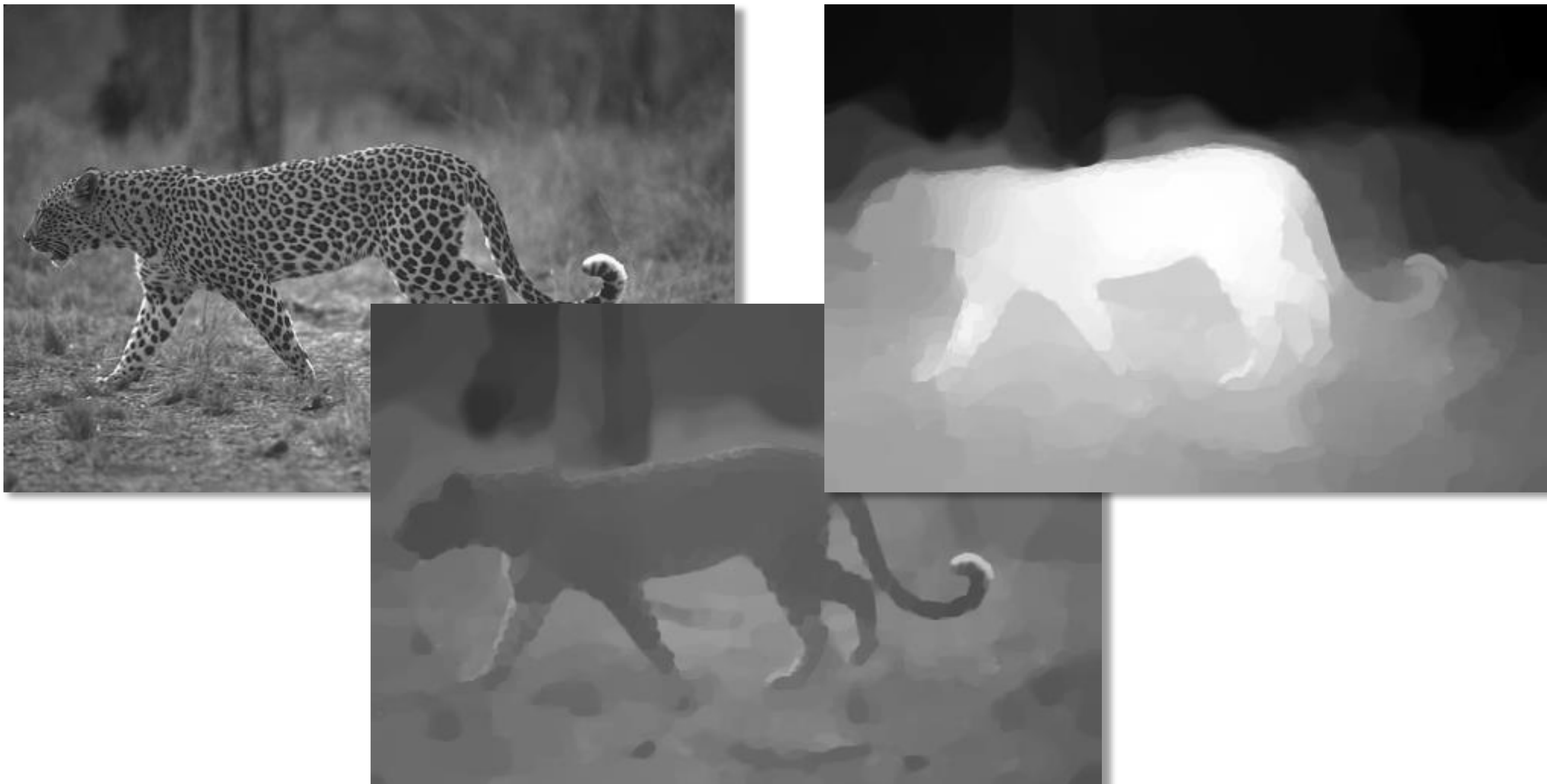
lifted precise
8 labels

Mean and variance estimation

Mean and variance estimation



Mean and variance estimation



Optical flow and image registration

Optical flow and image registration

Take-home

- Goal: global minimizer of nonconvex energies
- Lift into larger space
- Relax piecewise convex
- Much smaller problems, often 2-4 labels enough