Optimality of Relaxations for Integer-Constrained Problems

Jan Lellmann

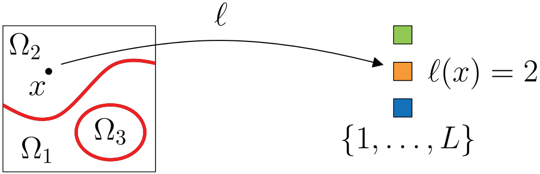
DAMTP, University of Cambridge

Joint work with: F. Lenzen, C. Schnörr (IPA/HCI, University of Heidelberg)

SIAM IS12, May 2012

Motivation – Problem

- Labeling problem:

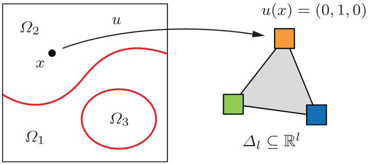


- Partition image domain Ω into L regions
- *Discrete* decision at each point in *continuous* domain Ω
- Variational Approach:

$$\min_{\ell} \underbrace{\int_{\Omega} s(\ell(x), x)dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

Model – Multi-Class Labeling

- Multi-class relaxation: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]



- Embed labels into $\mathbb{R}^L$ as $\mathcal{E} := \{e^1, \ldots, e^L\}$, relax integrality constraint to the unit simplex:

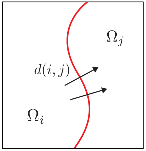
$$\Delta_L := \{x \in \mathbb{R}^L | x \geqslant 0, \sum_i x_i = 1\} = \operatorname{conv} \mathcal{E},$$

$$\min_{u \in \mathrm{BV}(\Omega, \Delta_L)} f(u), \quad f(u) := \int_\Omega \langle u(x), s(x) \rangle dx + \int_\Omega \Psi(Du)$$

- Advantages: No explicit parametrization, rotation invariance, *convex*

Model – Envelope Relaxation

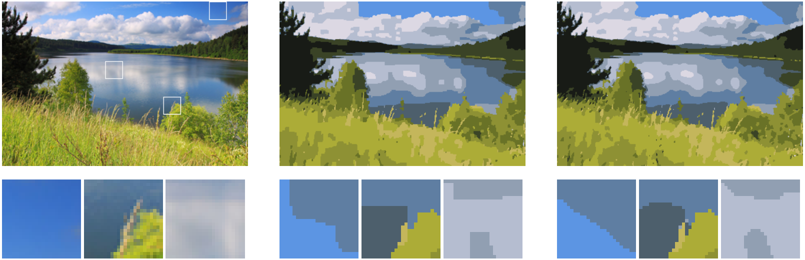
- $J(\ell)$: Weight boundary length by *interaction potential* $d(i,j)$



- ψ implicitly defined for given d [ChambolleCremersPock08,LellmannSchnoerr10] $\longrightarrow$ *metric labeling*.

Motivation – BV formulation

- Spatially continuous formulation avoids metrication artifacts:



Model – Rounding

- *Fractional* solutions may occur:

- **Goal:** Find *rounding scheme* $u^* \mapsto \bar{u}^* : \Omega \to \{e^1, \dots, e^L\}$ such that

$$f(\underbrace{\bar{u}^*}_{\text{rounded relaxed solution}}) \leqslant C f(\underbrace{u^*_{\mathcal{E}}}_{\text{best integral solution}}).$$

for some $C \geqslant 1$.

Rounding – Generalized Coarea Formula

- Two-class case: Generalized *coarea formula* [Strang83, ChanEsedogluNikolova06, Zach et al. 09, Olsson et al. 09]

$$f(u) \quad = \quad \int_0^1 f(\bar{u}_\gamma) d\gamma \,, \quad \bar{u}_\gamma := \begin{cases} e^1, & u_1(x) > \gamma, \\ e^2, & u_1(x) \leqslant \gamma. \end{cases}$$

- Also: *Choquet integral, Lovász extension, levelable function*,...

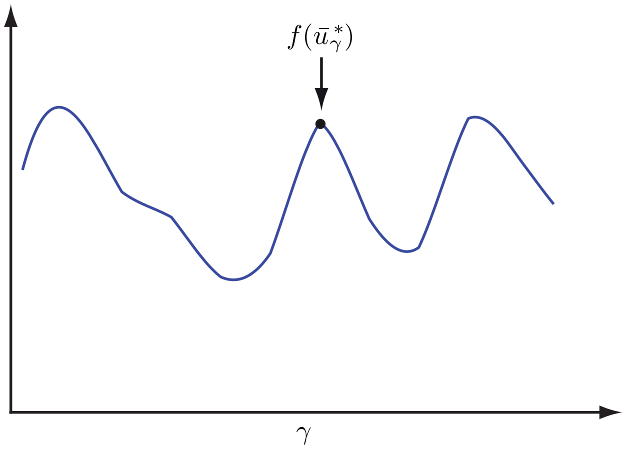
Rounding – Multi-Class Case

- *Probabilistic interpretation:*

$$f(u) \quad = \quad \int_0^1 f(\bar{u}_\gamma)d\gamma = \mathbb{E}_\gamma f(\bar{u}_\gamma)\,.$$

- Rounding step $u \mapsto \bar{u}_\gamma$ does not increase f *in the expectation*
- Consequence: global *integral* minimizer for a.e. γ! [ChanEsedogluNikolova06]

Rounding – Generalized Coarea Formula



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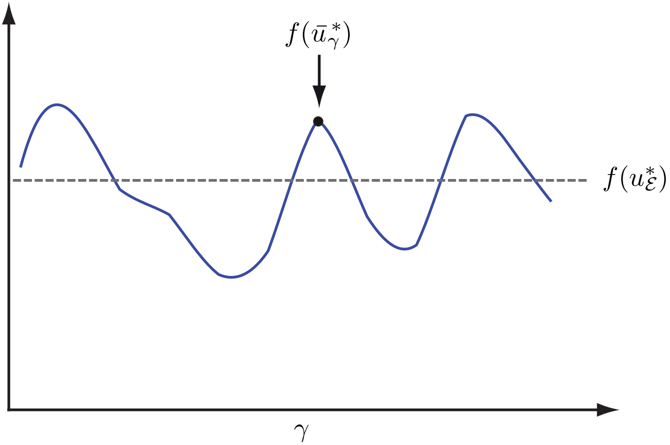

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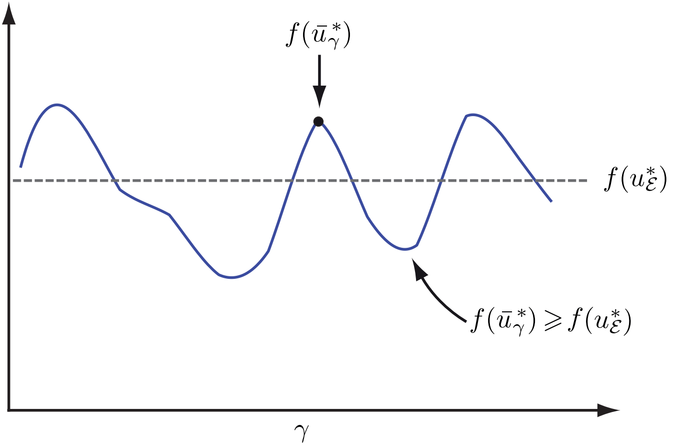

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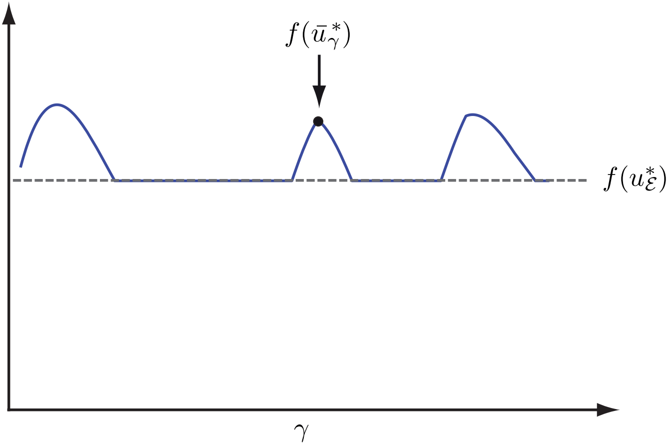

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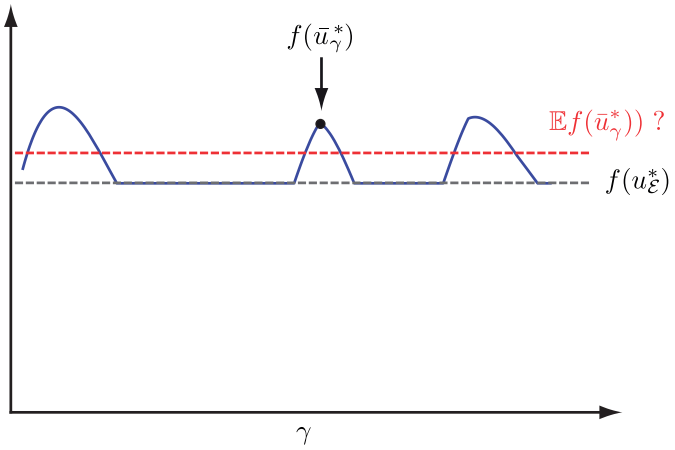

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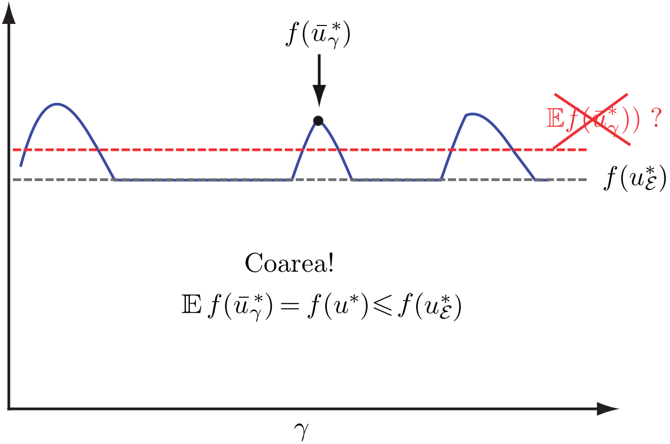

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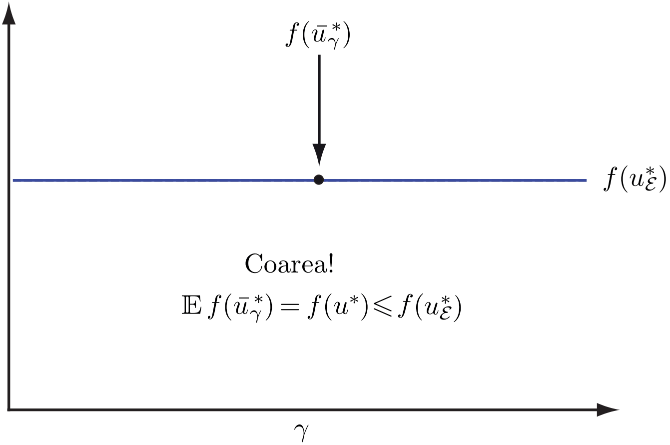

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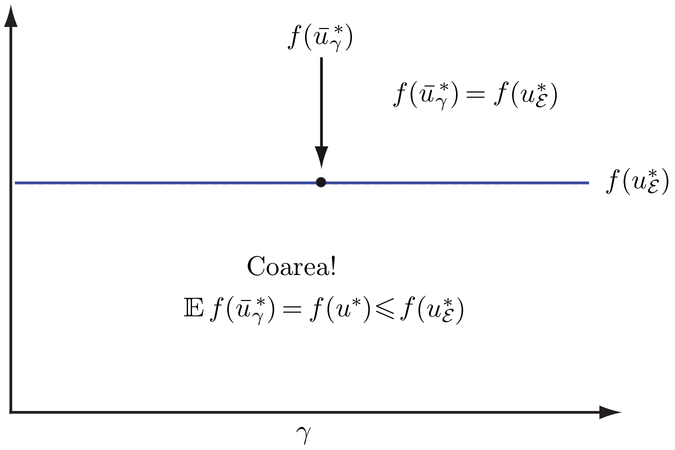


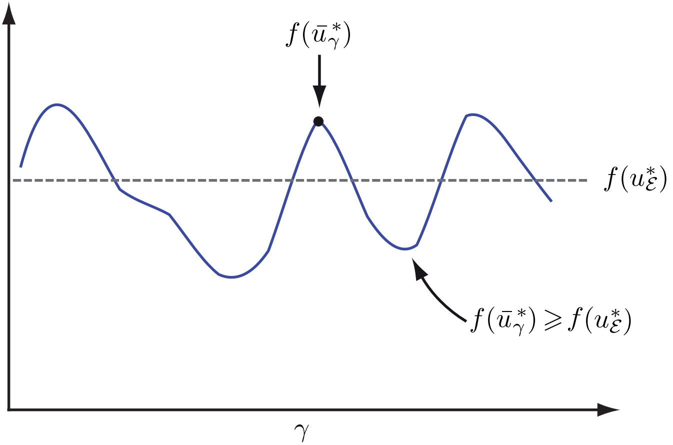
Rounding – Approximate Coarea Formula

- Multi-class generalization (*approximate* generalized coarea formula):

$$C f(u) \;\; \geqslant \;\; \int_{\Gamma} f(\bar{u}_{\gamma}) d\mu(\gamma) = \mathbb{E}_{\gamma} f(\bar{u}_{\gamma})$$

- Rounding step $u \mapsto \bar{u}_{\gamma}$ does not increase f too much *in the expectation*
- No bounds for *individual* γ!

Rounding – Approximate Coarea Formula



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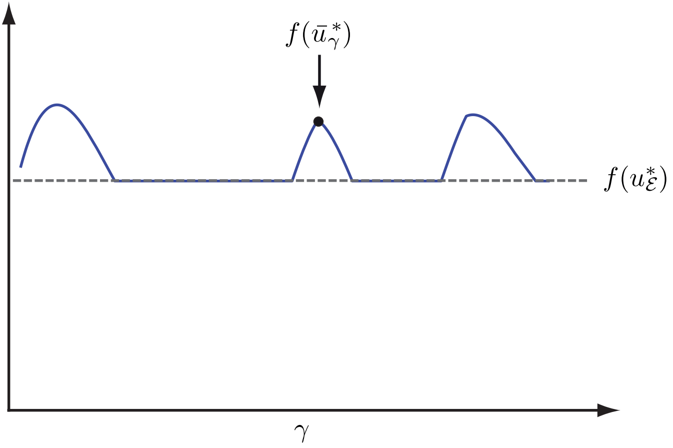

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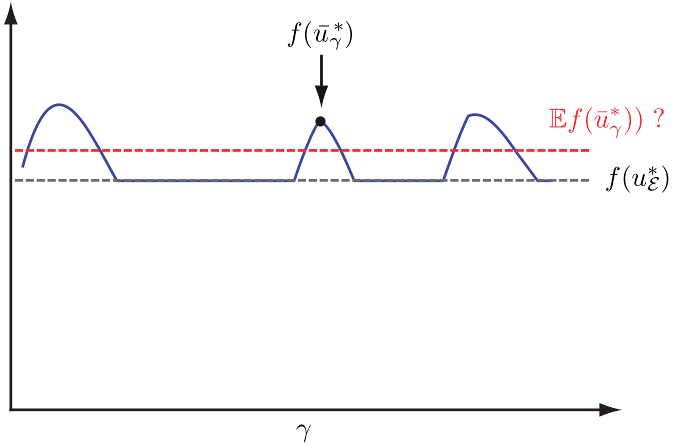

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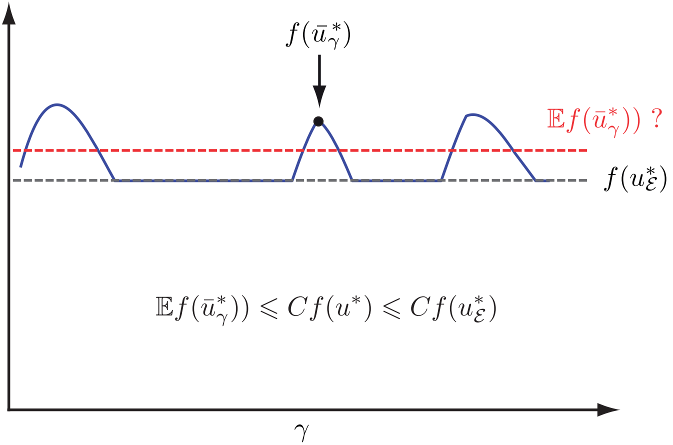
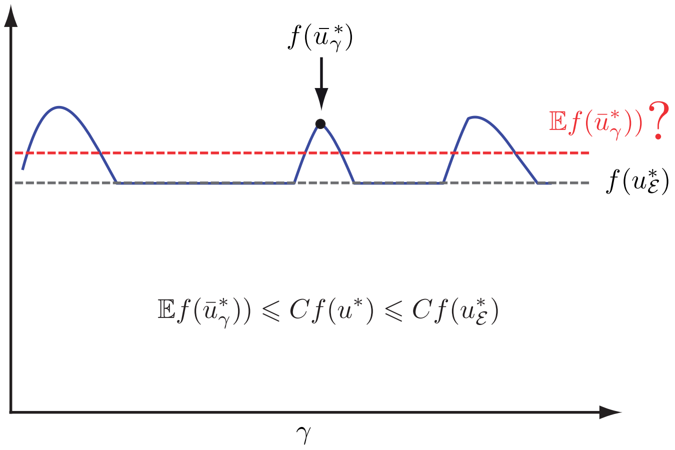


$$\mathbb{E}f(\bar{u}^*_\gamma)) \leqslant Cf(u^*) \leqslant Cf(u^*_\mathcal{E})$$

Rounding – Approximate Coarea Formula



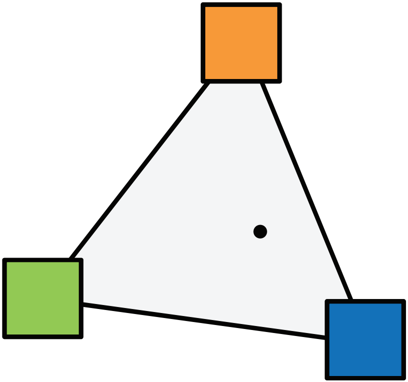
Rounding – Multi-Class Case

- Multi-class generalization (*approximate* generalized coarea formula):

$$C f(u) \quad \geqslant \quad \int_{\Gamma} f(\bar{u}_{\gamma}) d\mu(\gamma) = \mathbb{E}_{\gamma} f(\bar{u}_{\gamma})$$

- Need to define:
  - *parameter space* Γ
  - parametrized *rounding method* $u \mapsto \bar{u}_{\gamma}$
  - *probability measure* μ on Γ
  - bound C independent of input

Optimality – Example



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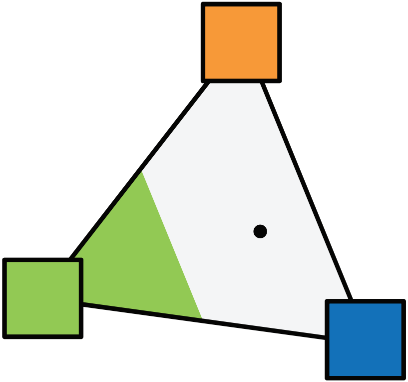

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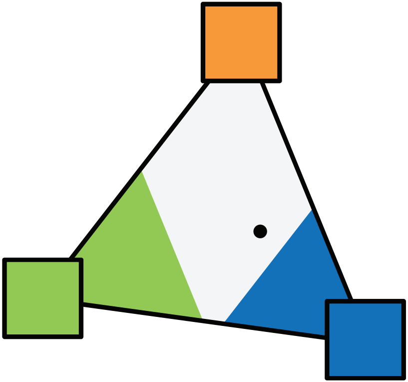

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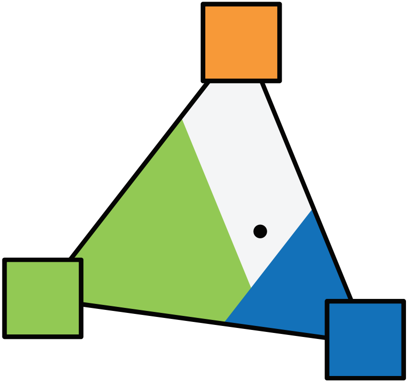

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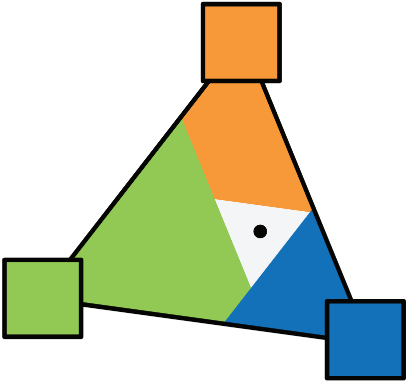

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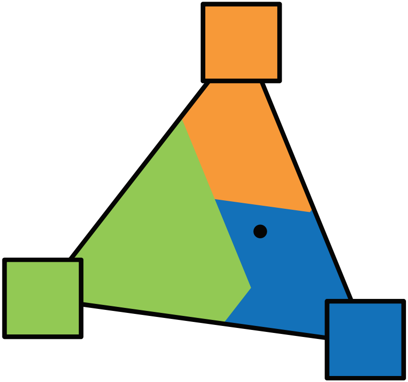

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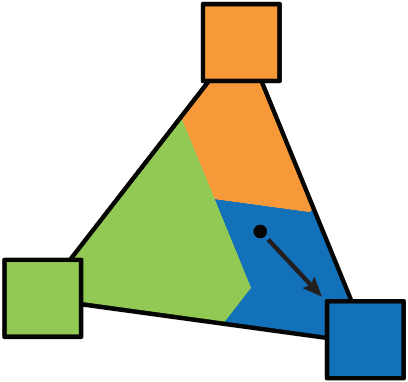


Optimality – Randomized Rounding

Algorithm 1 (Randomized Rounding in BV)

1. **Input:** $u^0 \in \mathrm{BV}(\Omega, \Delta_L)$
2. **For** $k = 1, 2, \ldots$
3. Sample $\gamma^k = (i^k, \alpha^k) \in \{1, \ldots, L\} \times [0, 1]$ uniformly
4. $u^k \leftarrow \begin{cases} e^{i^k}, & u^{k-1}_{i^k} > \alpha^k, \\ u^{k-1}, & u^{k-1}_{i^k} \leqslant \alpha^k. \end{cases}$
5. **Output:** Limit $\bar{u}$ of (u^k)

- Parameter space: *sequences* $\gamma \in \Gamma := (\{1, \ldots, L\} \times [0, 1])^{\mathbb{N}}$

Optimality – Termination

Theorem (Termination)

Let $u \in \mathrm{BV}(\Omega, \Delta_L)$. Then (almost surely) Alg. 1 generates a sequence that becomes stationary in some $\bar{u} \in \mathrm{BV}(\Omega, \mathcal{E})$.

- Result is in BV
- Independent of data term

Randomized Rounding – Motivation

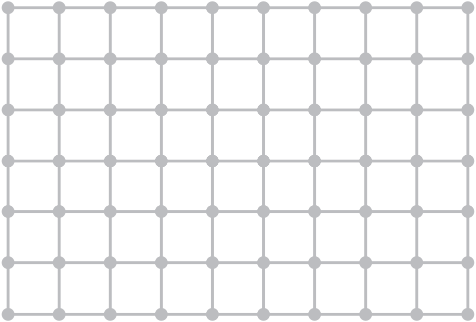
- In finite-dimensional setting: Can show for uniform metric [KleinbergTardos02]:

$$\mathbb{E}_\gamma f(\bar{u}_\gamma) \leqslant 2 f(u).$$

- Similar bounds:
  - multiway cut [Dahlhaus et al. 94]
  - α-expansion [Boykov et al. 01]
  - LP relaxation [KomodakisTziritas 07]

Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



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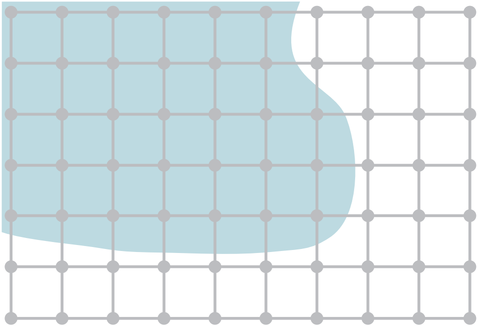

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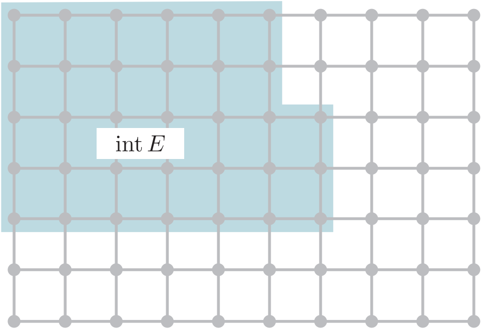

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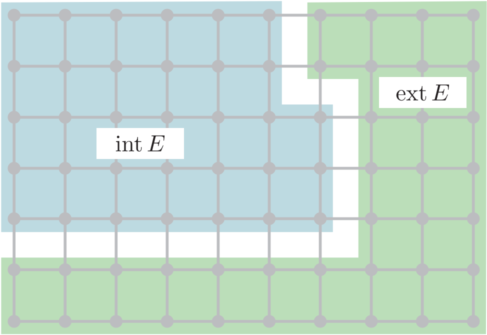

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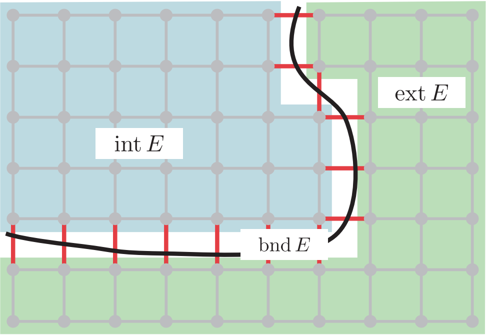

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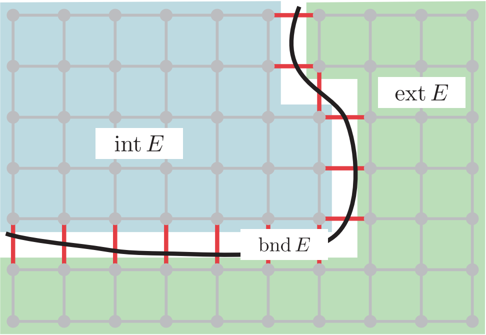
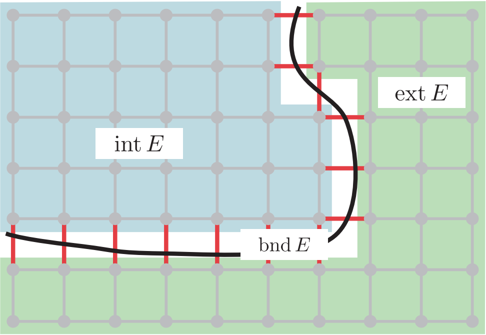


$$\mathbb{R}^n : \quad J(u) \leqslant J|_{\text{int } E}(u) + J|_{\text{ext } E}(u) + c \operatorname{Per}(E)$$

Optimality – Proof

$$E := \{x \in \Omega | u_{i^k} > \alpha^k\}$$



$$\mathbb{R}^n: \quad J(u) \leqslant J|_{\text{int}\,E}(u) + J|_{\text{ext}\,E}(u) + c\,\text{Per}(E)$$

$$\text{BV}(\Omega): |\psi(Du)|(\Omega) \leqslant |\psi(Du)|(E^1) + |\psi(Du)|(E^0) + c\,\text{Per}(E),$$

$$(E)^t := \{x \in \Omega | \lim_{\rho \searrow 0} \frac{|\mathcal{B}_\rho(x) \cap E|}{|\mathcal{B}_\rho(x)|} = t\}, \quad t \in [0,1].$$

Optimality – Main Result

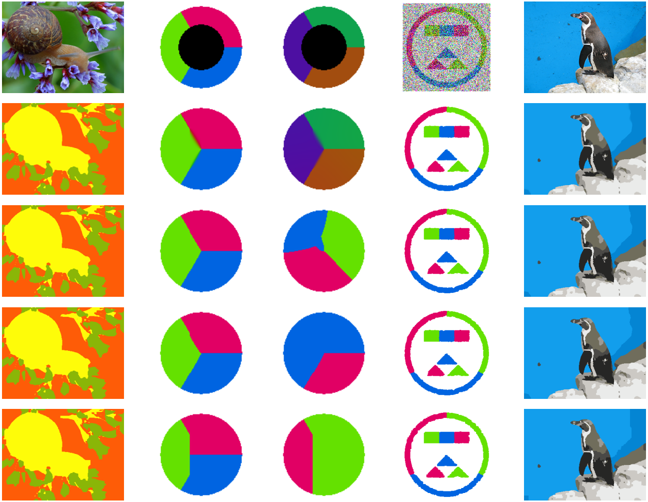
Theorem (Optimality [LellmannLenzenSchnoerr2011])

Let $u \in \mathrm{BV}(\Omega, \Delta_L)$*,* $s \in L^\infty(\Omega)^L$*,* $s \geqslant 0$*,* d *metric. Then*

$$\mathbb{E}f(\bar{u}) \leqslant 2\frac{\max_{i\neq j} d(i,j)}{\min_{i\neq j} d(i,j)} f(u) \qquad \textit{and} \qquad \mathbb{E}f(\bar{u}^*) \leqslant 2\frac{\max_{i\neq j} d(i,j)}{\min_{i\neq j} d(i,j)} f(u^*_{\mathcal{E}}).$$

- Provides "approximate" generalized coarea formula
- Compatible with bounds for finite-dimensional multiway cut, α-expansion, LP relaxation
- Formulated in BV, independent of discretization, true *a priori* bound

Experiments

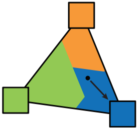


Experiments – Results

problem	1	2	3	4	5	6	7	8	9	10
# points	76800	14400	14400	129240	76800	86400	86400	76800	86400	110592
# labels	3	4	4	4	8	12	12	12	12	16
mean # iter.	7.27	7.9	8.05	10.79	31.85	49.1	49.4	49.4	49.7	66.1
a priori $C-1$	1.	1.	1.	1.	1.	1.	1.	1.	1.	2.6332
a posteriori										
- first-max	0.0007	0.0231	0.2360	0.0030	0.0099	0.0102	0.0090	0.0101	0.0183	0.0209
- prob. mean	0.0010	0.0314	0.1073	0.0045	0.0177	0.0195	0.0174	0.0219	0.0309	0.0487
- prob. best	0.0007	0.0231	0.0547	0.0029	0.0138	0.0152	0.0134	0.0155	0.0247	0.0292

Summary

- **Setting:**
  - Tight convex relaxation of multiclass image labeling
- **Bounds:**
  - *Probabilistic a priori* bound
  - Approximate generalized coarea formula
  - Compatible with finite-dimensional results



$$C f(u) \geqslant \int_{\Gamma} f(\bar{u}_{\gamma}) d\mu(\gamma) = \mathbb{E}_{\gamma} f(\bar{u}_{\gamma})$$

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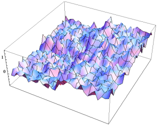
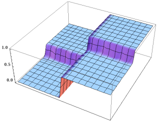
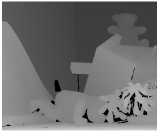
Motivation – Multiclass Labeling

- **Applications:** Denoising, segmentation, 3D reconstruction, depth from stereo, inpainting, photo montage, optical flow,...

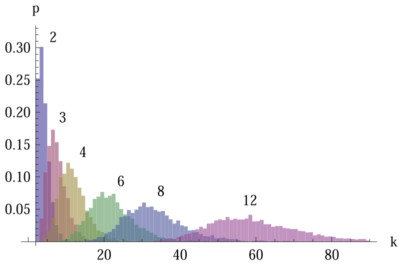
Bounds – A priori: Properties

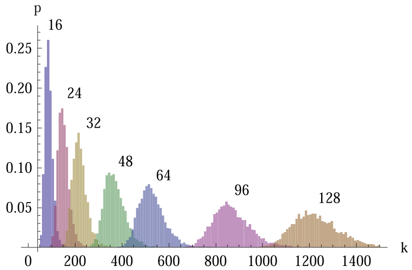
Definition

For some sequence (γ^k), if (u_γ^k) becomes stationary at some $u_\gamma^{k'} \in \mathbb{N}$, denote output $\bar{u}_\gamma := u_\gamma^{k'}$. For some functional $f : \mathrm{BV}(\Omega)^L \to \mathbb{R}$, define

$$
\begin{aligned}
f(\bar{u}_\gamma) : \quad & \Gamma^{\mathbb{N}} \to \mathbb{R} \cup \{+\infty\} \\
\gamma \in \Gamma^{\mathbb{N}} \quad & \mapsto f(\bar{u}_\gamma) := \begin{cases} f(u_\gamma^{k'}), & (u_\gamma^k) \text{ stationary at } u_\gamma^{k'} \in \mathrm{BV}(\Omega)^L, \\ +\infty, & \text{otherwise.} \end{cases} \qquad (*)
\end{aligned}
$$

Experiments – Iterations





Experiments II