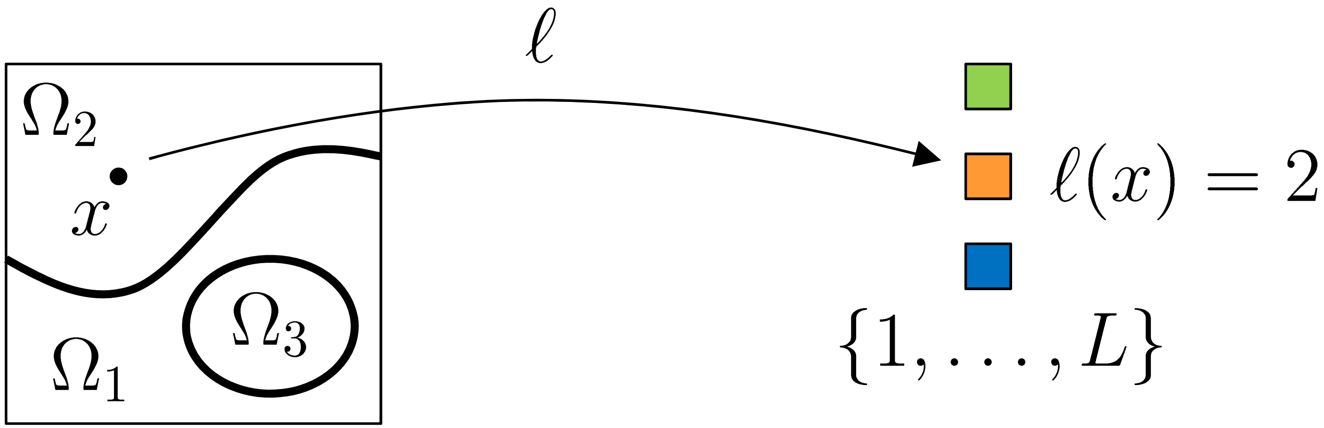
Convex Multi-Class Image Labeling by Simplex-Constrained Total Variation

J. Lellmann, J. Kappes,
F. Becker, J. Yuan, C. Schnörr

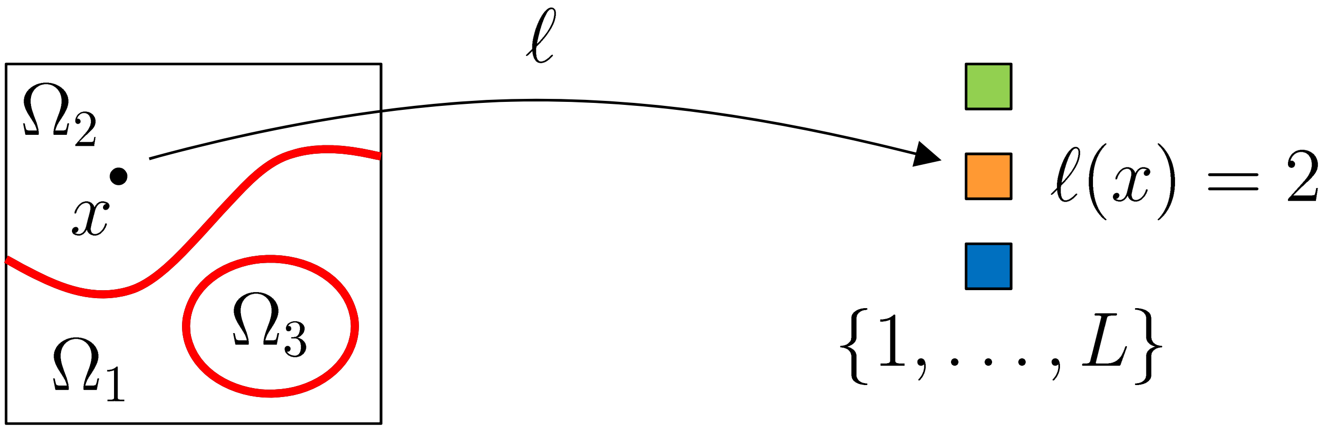
- Multi-Class Labeling



- Variational Approach:

$$\min_{\ell} \underbrace{-\int_{\Omega} s(\ell(x), x)dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

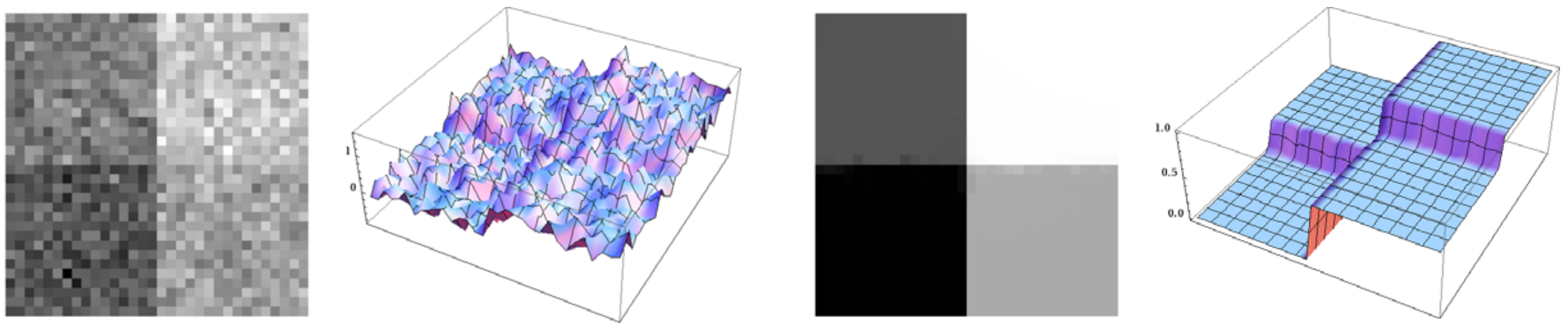
- Multi-Class Labeling



- Variational Approach:

$$\min_{\ell} - \underbrace{\int_{\Omega} s(\ell(x), x) dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

- Denoising/color segmentation



- Image segmentation
- Stereo matching
- Inpainting, photo montage etc.

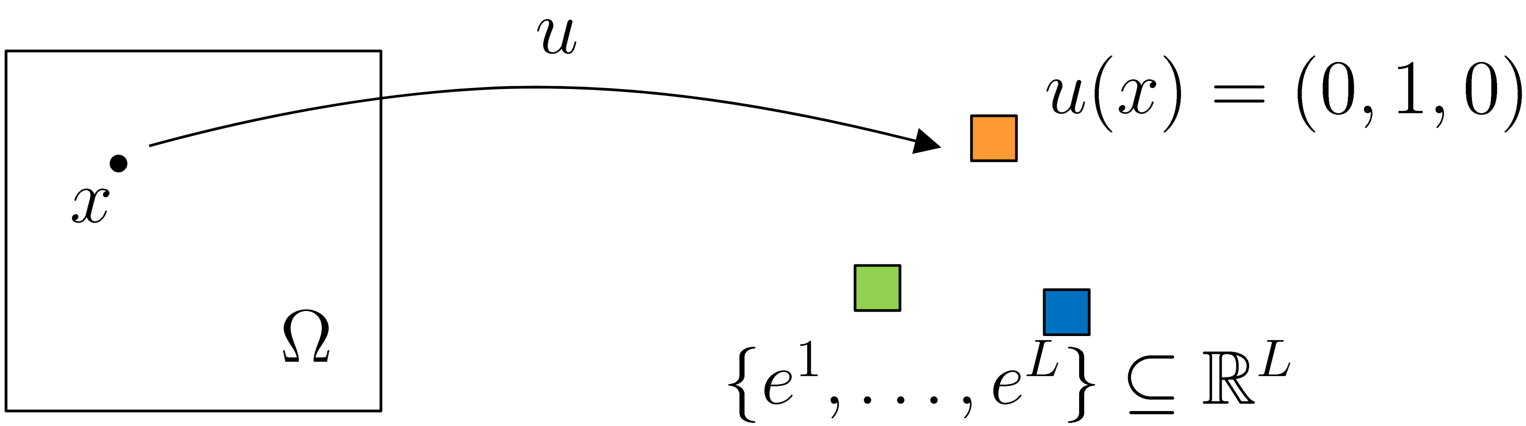
Related Work

	n-class	contin.	unordered
Graph Cuts	-	-	-
Boykov/Veksler/Zabih '01 Kleinberg/Tardos '02	x	-	x
Strang '81, Appleton/Talbot '05	-	x	-
Chan/Esedoglu/Nikolova '04	-	x	-
Lie/Lysaker/Tai '04	x	x	x
Zach et al. '08	x	x	x
Chambolle/Cremers/Pock '08	x	x	(x)
Proposed approach	x	x	x

- Continuous convex formulation:

$$\inf_{u:\Omega\to\mathbb{R}^n} f(u)\ , \qquad f(u) = -\int_\Omega \langle u(x), s(x)\rangle dx + \lambda\,\mathrm{TV_v}(u)\ , \qquad \lambda > 0$$

- Linearize data term

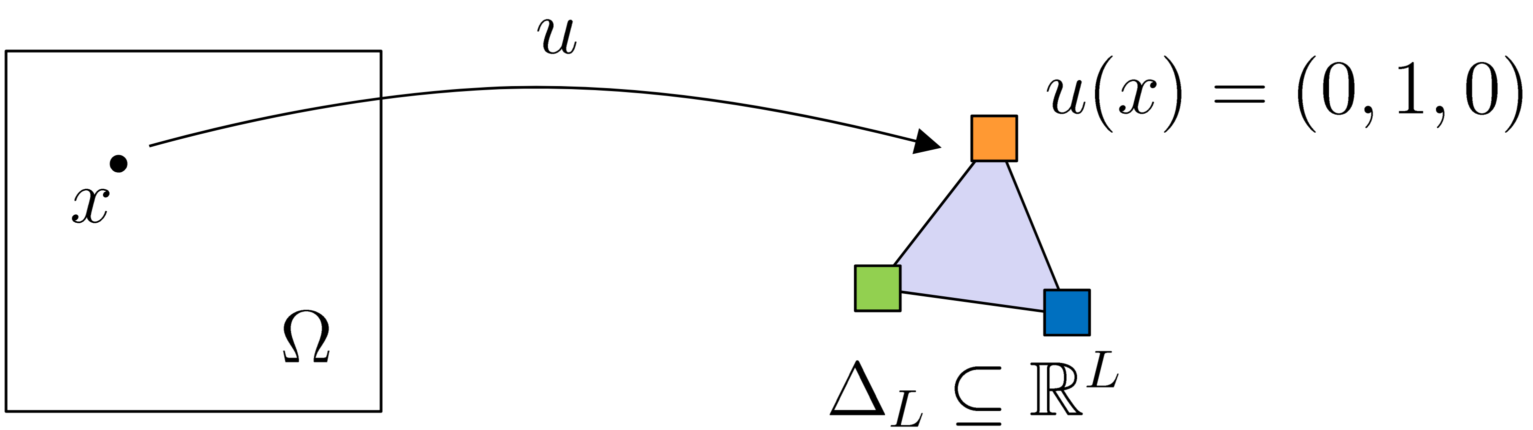


- Relax to the unit simplex – convex constraints

- Continuous convex formulation:

$$\inf_{u:\Omega\to\mathbb{R}^n} f(u)\ , \qquad f(u) = -\int_\Omega \langle u(x), s(x)\rangle dx + \lambda\, \mathrm{TV_v}(u)\ , \qquad \lambda > 0$$

- Linearize data term



- Relax to the unit simplex – convex constraints

- Continuous convex formulation:

$$\inf_{u \in C} f(u) , \qquad f(u) = -\int_\Omega \langle u(x), s(x) \rangle dx + \lambda \, \mathrm{TV}_\mathrm{v}(u) , \qquad \lambda > 0$$

- „Potts“ regularizer („MTV over labels“)

$$\mathrm{TV}_\mathrm{v}(u) = \int_\Omega \sqrt{\|\nabla u_1\|^2 + \cdots + \|\nabla u_L\|^2} dx$$

- No directional bias, no implicit ordering
- Overall convex

Outer problem – Douglas-Rachford Splitting

- Solve

$$u^* = \arg\min_u \underbrace{f_1(u)}_{f(u)} + \underbrace{f_2(u)}_{\delta_C(u)}$$

- Equivalently, find fixpoint of

$$\begin{aligned} u^k &= \arg\min_u \frac{1}{2}\|u - (z^k + \tau s)\|^2 + (\tau\lambda)\mathrm{TV}_{\mathrm{v}}(u)\,, \\ w^k &= \Pi_C(2u^k - z^k)\,, \\ z^{k+1} &= z^k + w^k - u^k\,. \end{aligned}$$

- Globally convergent under mild assumptions for any step size τ [e.g. Eckstein 1989]

Outer problem – Douglas-Rachford Splitting

- Solve

$$u^* = \arg\min_u \underbrace{f_1(u)}_{f(u)} + \underbrace{f_2(u)}_{\delta_C(u)}$$

- Equivalently, find fixpoint of

$$\begin{aligned} u^k &= \arg\min_u \frac{1}{2}\|u - (z^k + \tau s)\|^2 + (\tau\lambda)\mathrm{TV}_{\mathrm{v}}(u)\,, \\ w^k &= \underline{\Pi_C(2u^k - z^k)}\,, \\ z^{k+1} &= z^k + w^k - u^k\,. \end{aligned}$$

- Globally convergent under mild assumptions for any step size τ [e.g. Eckstein 1989]

Outer problem – Douglas-Rachford Splitting

- Solve

$$u^* = \arg\min_u \underbrace{f_1(u)}_{f(u)} + \underbrace{f_2(u)}_{\delta_C(u)}$$

- Equivalently, find fixpoint of

$$\begin{aligned} u^k &= \underline{\arg\min_u \frac{1}{2}\|u-(z^k+\tau s)\|^2 + (\tau\lambda)\mathrm{TV_v}(u)}\,, \\ w^k &= \Pi_C(2u^k - z^k)\,, \\ z^{k+1} &= z^k + w^k - u^k\,. \end{aligned}$$

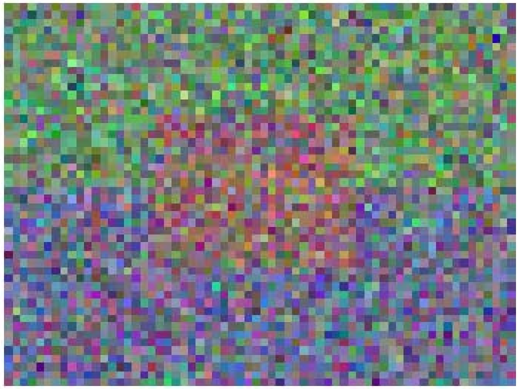
- Globally convergent under mild assumptions for any step size τ [e.g. Eckstein 1989]

Inner Problem – Dual FB Approach

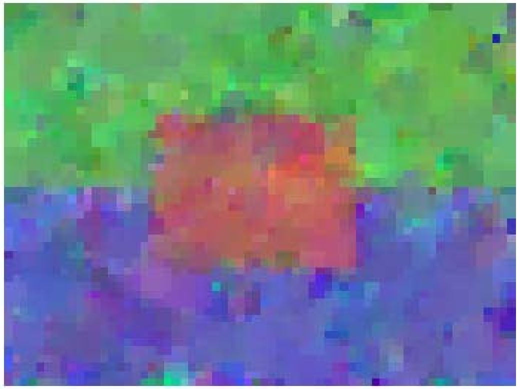
- Inner problem:

$$u^k = \arg\min_u \frac{1}{2}\|u - (z^k + \tau s)\|^2 + (\tau\lambda)\mathrm{TV_v}(u)$$

- Rudin-Osher-Fatemi – like denoising



original input



after first outer iteration

Inner Problem – Dual FB Approach

- Inner problem:

$$u^k = \arg\min_u \frac{1}{2}\|u - (z^k + \tau s)\|^2 + (\tau\lambda)\mathrm{TV_v}(u)$$

- Fixpoint (Gradient-Projection on dual problem) [cf. Chambolle05, Bresson07, Duval08]
  - Simple steps
  - Convergent
  - Can we do better?

Inner Problem – Half-Quadratic Approach

- Inner problem:

$$u^k = \arg\min_u \frac{1}{2}\|u - (z^k + \tau s)\|^2 + (\tau\lambda)\mathrm{TV_v}(u)$$

- Half-Quadratic (Geman/Reynolds '92) [Yang08]:

$$\min_{u,y} \frac{1}{2}\|u - (z^k + \tau s)\|^2 + \underline{\frac{\beta}{2}(\tau\lambda)\|y - \nabla u\|^2} + (\tau\lambda)\|y\|$$

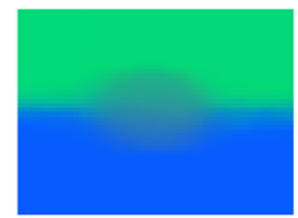
  - Still ϵ - suboptimal for $\beta \geqslant n/(2\epsilon)$
  - Solve for y: Soft thresholding, separable
  - Solve for u: Linear equations, fast using DCT

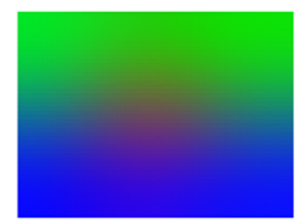
- Half-Quadratic:

$$\min_{u,y} \frac{1}{2}\|u - (z^k + \tau s)\|^2 + \frac{\beta}{2}(\tau\lambda)\|y - \nabla u\|^2 + (\tau\lambda)\|y\|$$

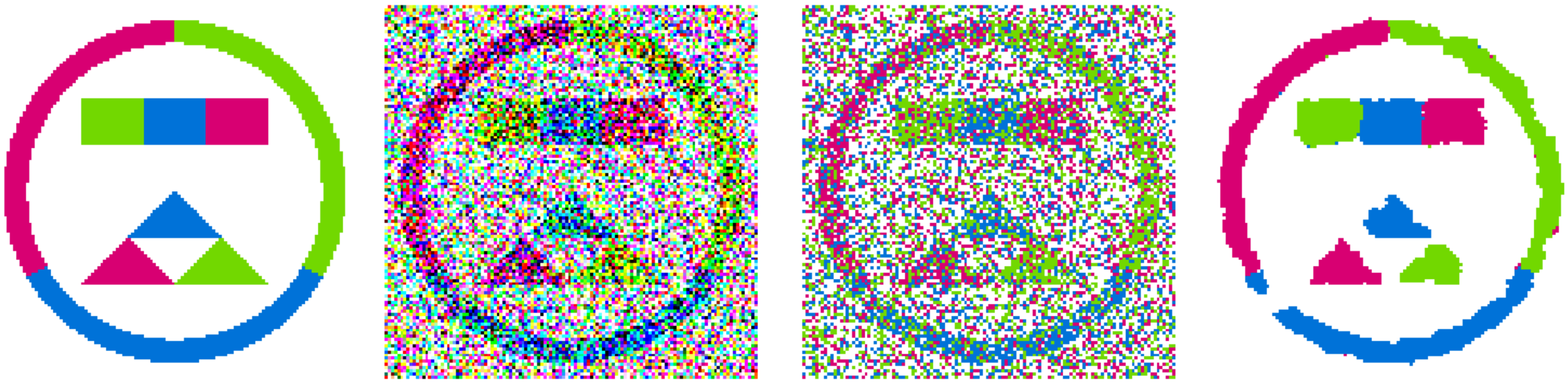
  - Speed depends on penalty parameter schedule
  - Fastest: increase β at each step

- Comparison:
  - Large $\tau\lambda$: HQ method wins.
  - Small $\tau\lambda$: Fixpoint method wins.
  - Runtime factors ~4-5



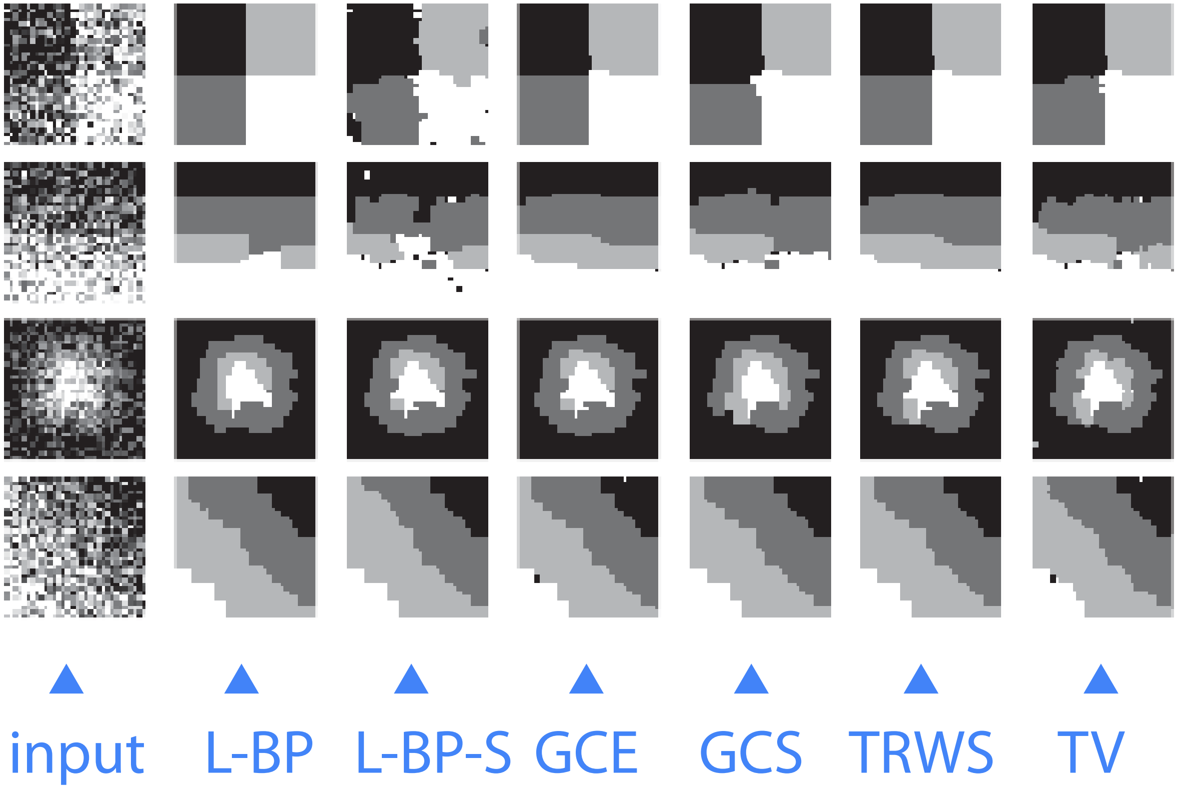


- 4-class color segmentation:

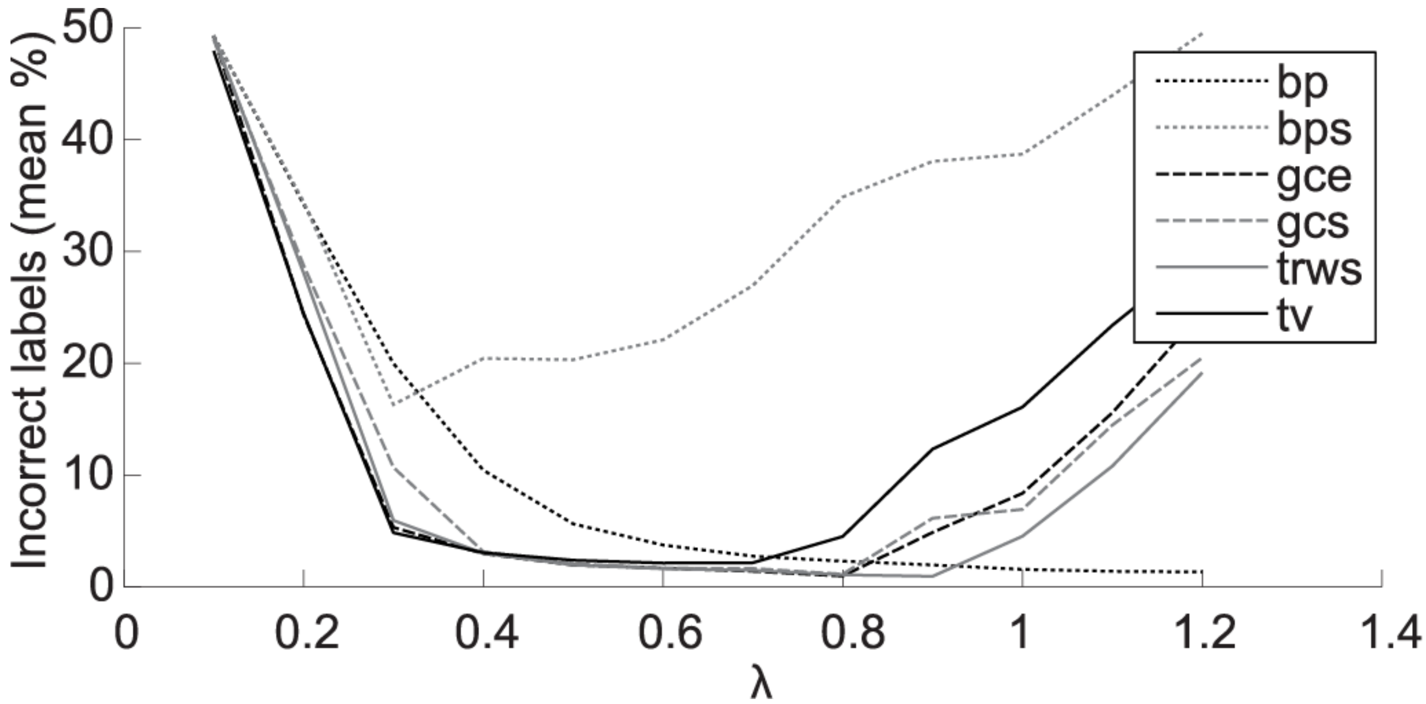


- L1 distance
- ~20 outer iterations

- Grayscale segmentation – comparison



- Error percentage over 20 experiments each



- Conclusion:
  - Discrete combinatorial $\rightarrow$ continuous convex
  - Douglas-Rachford + Fixpoint/Half-Quadratic
  - Globally convergent, close to discrete approaches
- Future:
  - Theoretical bounds
  - More involved regularizers