On Metric Image Labeling and Sparsity

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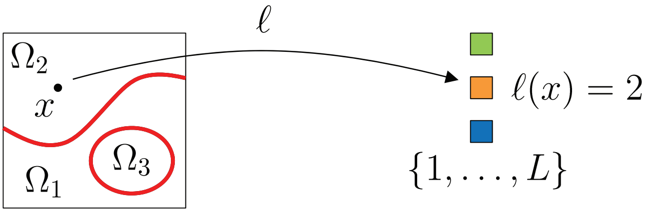
Problem

- How to extend this approach to more than 2 labels?
- Multi-Class Labeling

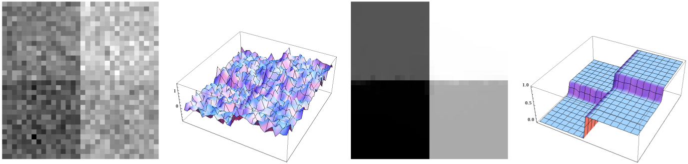


- Variational Approach:

$$\min_{\ell} \underbrace{\int_{\Omega} s(\ell(x), x)dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

Applications

- Denoising/color segmentation

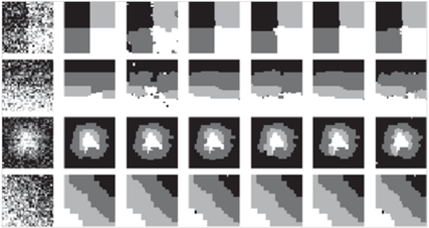


- Image segmentation
- Stereo matching

- Inpainting, photo montage, etc...

Motivation

- Discrete graph cut based methods have inherent grid bias.
- Many multi-label algorithms find only local minimum:



(input, L-BP, L-BP-S, GCE, GCS, TRWS, TV)

- Can do as well by solving single convex problem with finite difference discretization (no grid bias)

Related Work

- Alberti,Bouchitté,dal Maso 1999, Ambrosio,Braides 1990: Continuous background
- Boykov et al. 2001: On graph; reduce to multiple graph cuts by $\alpha-$expansion, optimality bound
- Kleinberg, Tardos 2002: On graph; LP relaxation, optimality bound
- Zach et al. 2008; Lellmann et al. 2008: General approach for Potts model; Arrow-Hurwicz / Douglas-Rachford
- Chambolle/Cremers/Pock 2008: Approximate envelope of discrete problem for nonstandard potentials, Arrow-Hurwicz optimization
- Lie, Lysaker, Tai 2004; Bae, Tai 2008: Differently ordered labels, additional term to enforce binarity, Augmented Lagrangian

Approach – Questions

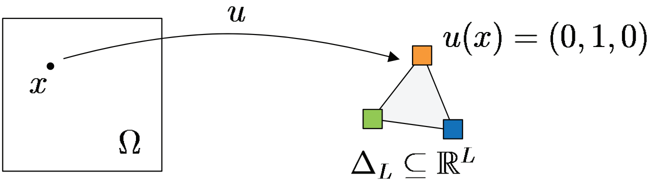
- Variational Approach:

$$\min_{\ell} \underbrace{\int_{\Omega} s(\ell(x), x)dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

- Questions:
  - How to formulate as a (convex) continuous problem?
  - How to extend data term and regularizer on convex feasible set?
  - How to numerically solve the resulting nonsmooth problem?

Approach – Relaxation

- Embed labels into $\mathbb{R}^L$ as $\{e^1, \ldots, e^L\}$.
- Relax to the unit simplex:



- Continuous convex formulation:

$$\min_{u:\Omega\to\Delta_L} f(u)\,, \qquad f(u) = \int_\Omega \langle u(x), s(x)\rangle dx + J(u)$$

- Linear data term
- Convex constraints

Regularizer – How to extend?

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- Simplest Case: Boundary length (*Potts*) $\leftrightarrow$ *vector* Total Variation:

$$\mathrm{TV}(u) = \int_\Omega \sqrt{\|Du_1\|^2 + \cdots + \|Du_L\|^2} dx$$
$$= \sup_{v \in C_c^\infty(\Omega, \mathbb{R}^{l*d})} \left\{ \int_\Omega \langle u, \mathrm{Div}\, v \rangle dx \,\middle|\, \|v(x)\| \leqslant 1, x \in \Omega \right\}$$

- General Approach:

$$J(u) = \sup_{v \in C_c^\infty(\Omega, \mathbb{R}^{l*d})} \left\{ \int_\Omega \langle Au, \mathrm{Div}\, v \rangle dx \,\middle|\, v(x) \in \mathcal{D}, x \in \Omega \right\}$$

- $\Rightarrow$ *Nonsmooth* convex problem
- Minimizer exists in BV.

Regularizer – Non-uniform potentials

- Vary *interaction potentials*, i.e. penalize perimeter according to weight

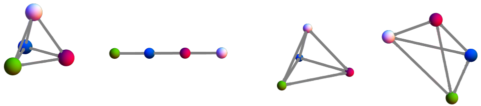
$$w_{ij}$$

depending on the *labels* i, j of the adjacent regions:

- Can show that $d(i, j) := w_{ij}$ must be a metric if J convex.

Regularizer – Embedding Approach

- Embedding [LS09] approach means linear embedding in *label space*.



- Euclidean interaction potentials can be represented exactly:

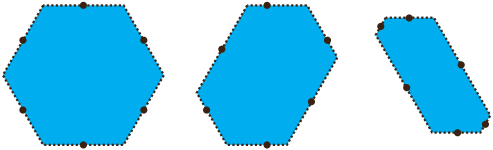
$$w_{ij} = \|a^i - a^j\|_2$$

- Non-Euclidean interaction potentials can be approximated using SDP.

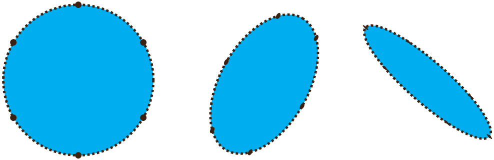
$$\begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix} \quad vs. \quad \begin{pmatrix} 0 & 1.15 & 1.92 & 2.08 \\ 1.15 & 0 & 1.15 & 1.92 \\ 1.92 & 1.15 & 0 & 1.15 \\ 2.08 & 1.92 & 1.15 & 0 \end{pmatrix}$$

Regularizer – Varying $\mathcal{D}$ (tightness)

- Local envelope [CCP08]: Tight, $\mathcal{D}$ complicated (inexact projection)



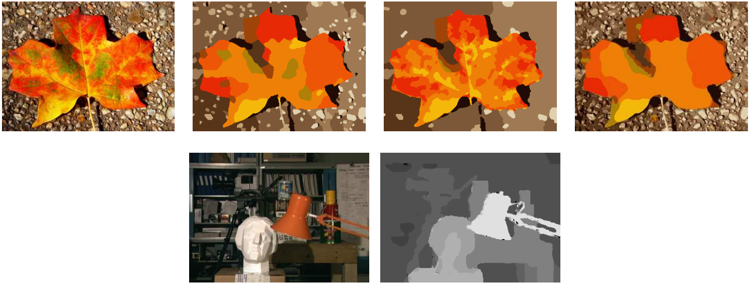
- Embedding [LS09]: Not as tight, now $\mathcal{D}$ is trivial (unit sphere)



- $\Rightarrow$ Trade-off between tightness of relaxation and computational effort.

Regularizer – Embedding Approach

- Euclidean interaction potentials are still powerful:



Optimization – Problem

- Overall Problem:

$$\min_{u:\Omega\to\Delta_L} \int_\Omega \langle u(x), s(x)\rangle dx + \sup_{v\in\mathcal{D}} \int_\Omega \langle Au, \mathrm{Div}\, v\rangle dx$$

- Convex but nonsmooth problem with constraints
- Explicit smoothing approaches: critical due to properties of desired solution
- Here: First order primal/dual methods: small memory footprint, easy to parallelize

Optimization – Douglas-Rachford Splitting

- Split into objective and constraints:

$$u^* = \arg\min_u \underbrace{f_1(u)}_{f(u)} + \underbrace{f_2(u)}_{\delta_C(u)}$$

- Apply Douglas-Rachford Splitting: Iterate to find fixpoint of

$$\begin{aligned} u^k &= \arg\min_u \frac{1}{2}\|u - (z^k + \tau s)\|^2 + \tau J(u)\,, \\ w^k &= \Pi_C(2u^k - z^k)\,, \\ z^{k+1} &= z^k + w^k - u^k\,. \end{aligned}$$

- Globally convergent under mild assumptions for any step size τ [e.g. Eckstein1989].
- Related to Alternating Split Bregman, Alternating Direct Method of Multipliers [Setzer09].
- Solve inner "ROF" type denoising problems (strictly convex!) using dual fixpoint method [Chambolle05] or Half-Quadratic approach [Yang08] [Lellmann09].

Optimization – Nesterov approach

- Rewrite as bilinear saddle point problem:

$$\min_{u \in \mathcal{C}} \max_{v \in \mathcal{D}} g\,(u, v)\,,$$
$$g(u, v) := \langle u, s\rangle + \langle Lu, v\rangle - \langle b, v\rangle\,.$$

- Controlled smoothing by prox-function combined with fast smooth first-order method [Nesterov04].
- ε-optimal solution in $O(1/\varepsilon)$.
- Explicit suboptimality bounds with certificate (dual feasible point).
- No parameters besides desired suboptimality.

Optimization – Related methods

- Interior Point (SOCP): Large overhead, scales badly, memory footprint
- Appleton,Talbot 2005; Chambolle,Cremers,Pock 2008: Arrow-Hurwicz – have to choose step size
- Trobin,Pock,Cremers,Bischof 2008; Olsson 2009: Repeated binary fusion – cf. α-expansion, series of binary problems
- Pock,Schoenemann,Graber,Bischof,Cremers 2009: Popov-like primal/dual – now proven to converge, fast
- Lie,Lysaker,Tai 2004: Augmented Lagrangian, Uzawa

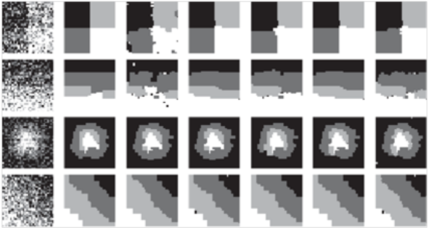
Experiments – Relaxation

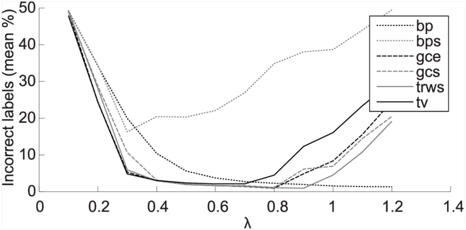
- Relaxation yields non-discrete results – threshold if desired

- Thresholding does not preserve optimality as in two-class case!
- No artifacts as with graph-based methods:

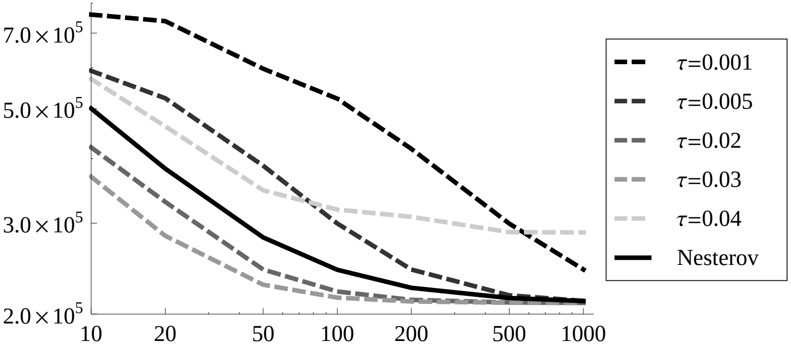
Experiments – Discretization

- Competes with graph cut-based methods:



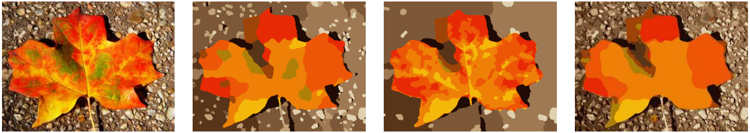


Experiments – Algorithm



Intermediate Summary

- Formulate *multi-class* labeling as *continuous convex* problem.
- Regularizer as support function
  - Trade-off: Complexity of dual constraint set vs. tightness of relaxation
  - Linear embedding permits many useful potentials at no extra cost



- Optimization:
  - Douglas-Rachford splitting: Expensive inner problems
  - Nesterov: Fast, parameter-free, explicit performance guarantees

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