Solution-driven Adaptive Total Variation Regularization

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Inverse Problems for Image Restoration

Variational approach

$$\min_u \quad \mathcal{S}(u, f) + \alpha \mathcal{R}(u)$$

with

$$\begin{array}{lcl} \mathcal{S} & : & \text{data term depending on data } f \\ \mathcal{R} & : & \text{regularization term} \\ \alpha & : & \text{regularization parameter} \end{array}$$

Example: total variation

$$\mathcal{R}(u) = \mathrm{TV}(u) = \int_\Omega \|\nabla u(x)\|_2 \, dx$$

Types of Regularization

Adaptive Regularization

penalization of ∇u (or higher derivatives) varies locally

- data-driven: adaptivity depends on input data f or guidance image u_0
- solution-driven: adaptivity is steered by the unknown u itself

Anisotropic Regularization

directionally dependent penalization of ∇u (or higher derivatives)

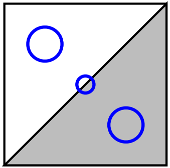
example:

$$\mathrm{TV}_a(u) := \int_\Omega \|\nabla u(x)\|_1 \, dx = \int_\Omega |\partial_x u(x)| + |\partial_y u(x)| \, dx$$

Prototypes of Adaptive TV

1. weighted TV:

$$\mathcal{R}(u) := \int_\Omega \alpha(x, v) \|\nabla u(x)\|_2 \, dx$$



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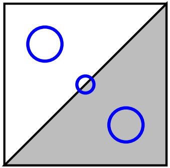
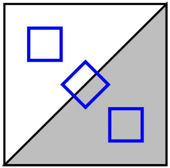


2. anisotropic TV I:

$$\mathcal{R}(u) := \int_\Omega \|R(x, v) \nabla u(x)\|_1 \, dx$$

with rotation matrix $R(x, v)$



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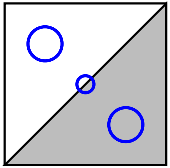

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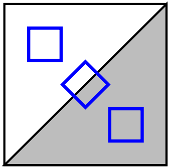
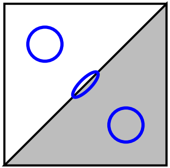


3. anisotropic TV II:

$$\mathcal{R}(u) := \int_\Omega \sqrt{\nabla u(x)^\top A(x,v)\nabla u(x)} \, dx$$

with symmetric positive-definite matrix $A(x)$



variable v: either f, a guidance image u_0 or u.

Related work I

1. Esedoglu & Osher,
   Decomposition of images by the anisotropic Rudin-Osher-Fatemi model, CPAA, 2004
2. Steidl & Teuber,
   Anisotropic smoothing using double orientations, SSVM, 2009
3. Grasmair,
   Locally Adaptive Total Variation Regularization, SSVM 2009
4. Dong et al.,
   Automated Regularization Parameter Selection in Multi-Scale Total Variation Models for Image Restoration, IJCV, 2011
5. Bayram & Kamasak,
   Directional total variation, IEEE Signal Proc. Letters, 2012

Related work II

6. Lefkimmiatis, Roussos et al., Convex generalizations of total variation based on the structure tensor, 2015

$$\mathcal{R}(u) = \int_\Omega \|(\sqrt{\lambda_1(u)}, \sqrt{\lambda_2(u)})^\top\|_p \, dx$$

where $\lambda_1(u), \lambda_2(u)$ are the eigenvalues of the structure tensor $J(u)$.

7. Åström et al., A Tensor Variational Formulation of Gradient Energy Total Variation, 2015

$$\mathcal{R}(u) = \int_\Omega \frac{\nabla u^\top A(u) \nabla u}{\|\nabla u\|_2} \, dx$$

8. Estellers et al., Adaptive Regularization with the Structure Tensor, 2015

$$\mathcal{R}(u) = \mathcal{R}_{J_k}(u)$$

where J_k is an iteratively updated estimate of the structure tensor.

General Definition

For $u \in L^1(\Omega)$

$$\mathcal{R}(u) := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx \mid \varphi \in C_c^1(\Omega; \mathbb{R}^n),\ \varphi(x) \in \mathcal{D}(x) \right\}$$

with *local constraint sets* $\mathcal{D}(x)$:

- closed
- convex
- bounded
- non-empty (containing $B_c(0)$ for some $c > 0$)

Higher order TV: exchange operator div.

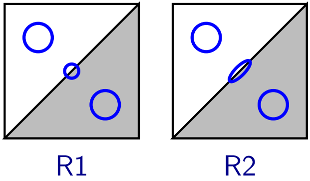
Our approach

For some arbitrary image v define

$$\mathcal{R}_v(u) := \sup\{\int_\Omega u \operatorname{div}\varphi \, dx \mid \varphi \in C_c^1(\Omega;\mathbb{R}^n),\ \varphi(x) \in \mathcal{D}(x,v)\}.$$

Examples for $\mathcal{D}(x,v)$:

- R1: balls with radius $\alpha(x,v)$ depending on $\|\nabla v\|_2$.
- R2: ellipses oriented/scaled w.r.t. the eigenvectors/eigenvalues of structure tensor $J(v)$.



Our approach

Search for a fix point of

$$v \to T(v) := \arg\min_u \mathcal{S}(u, f) + \mathcal{R}_v(u).$$

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After having found a fixed point u^*, we have

$$u^* = \arg\min_{u} \mathcal{S}(u, f) + \mathcal{R}_{u^*}(u).$$

$\leadsto$ solution-driven adaptive total variation regularization.

Theory

Under sufficient conditions

Continuous case: existence of a fixed point for a sub-class of $\mathcal{R}_v$'s,
The proof uses the Himmelberg fixed point theorem.

Discrete case: existence and uniqueness of a fixed point

theory is via quasi-variational inequality problems:

$$\begin{aligned}&\text{find } p^* \in \mathcal{D}(p^*)\\ &\langle \nabla_h (f - \operatorname{div}_h p^*), p - p^* \rangle \geq 0 \qquad \forall p \in \mathcal{D}(p^*)\end{aligned}$$

Details see: F. L., Adaptive Variational Regularization Techniques in Image Processing and Computer Vision. Habilitation Thesis, University of Lübeck, 2015

Results – Denoising

noisy image

SSIM[1] index:

R1 data-driven

0.794

→

R1 solution-driven

0.817

standard TV

0.806

R2 data-driven

0.711

→

R2 solution-driven

0.829

(sub-optimal parameters)

[1]Wang et al. "Image quality assessment: from error visibility to structural similarity.", 2004

Results – Deblurring

blurry image

SSIM [2] index:

R1 data-driven

0.832

→

R1 solution-driven

0.875

standard TV

0.835

R2 data-driven

0.834

→

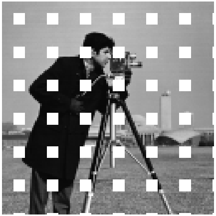
R2 solution-driven

0.871

(optimal parameters)

[2] Wang et al. "Image quality assessment: from error visibility to structural similarity.", 2004

Results – Inpainting



blurry image
SSIM index:

standard TV
0.969

R2 data-driven
0.968

→

R2 solution-driven
0.977

(optimal parameters)

Conclusion

- Increasing use of adaptive regularization techniques.
- Solution-driven adaptivity is preferable.
- Theoretical issues in general: function spaces, (non)-convexity, existence & uniqueness
- Advantage of our approach: inner problem is convex and well-posed & theory on fixed points.

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Himmelberg Fixed Point Theorem

Theorem (Theorem 2.3 in Agarwal and O'Regan, 2002)

Let E be a Banach space and let C be a closed convex subset of E. Then any weakly compact, weakly sequentially upper-semicontinuous map $F : C \to K(C)$ has a fixed point.

Proof utilizes

Theorem (Himmelberg Fixed Point Theorem)

Let T be a nonvoid convex subset of a separated locally convex space L. Let $F : T \to T$ be a u.s.c. multimap such that $F(x)$ is closed and convex for all $x \in T$, and $F(T)$ is contained in some compact subset C of T. Then F has a fixed point.

Results – A posteriori distribution