Anisotropic Third-Order Regularization for Sparse Digital Elevation Maps

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Acknowledgments: A. Bertozzi and A. Chen

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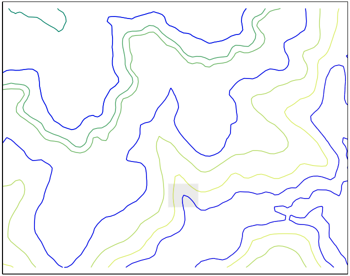
Surface interpolation

- *Input:* (parts of) contour lines
- *Output:* dense surface

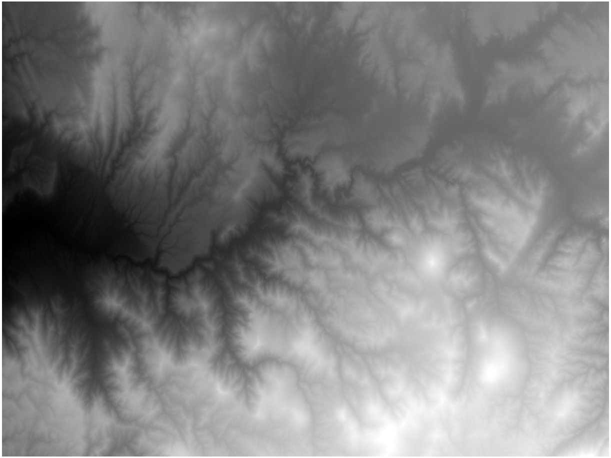


- Variational approach:

$$\min_{u:\Omega\to\mathbb{R}} R(u), \quad \text{s.t. } u(x) = u_0(x) \text{ for } x \in C.$$

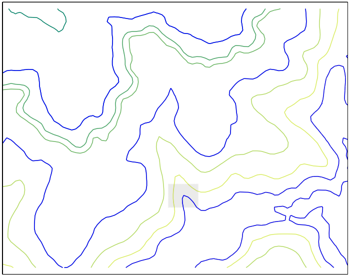
Digital elevation map

- Standard regularizers highly unlikely to work



Challenges – 1

- Challenges:
  - contour lines can have non-differentiabilities/high curvature, these are *features* and should be preserved!
  - data can be irregular/sparse – regularizer is much more important
  - discretization has large influence (also: boundary conditions)

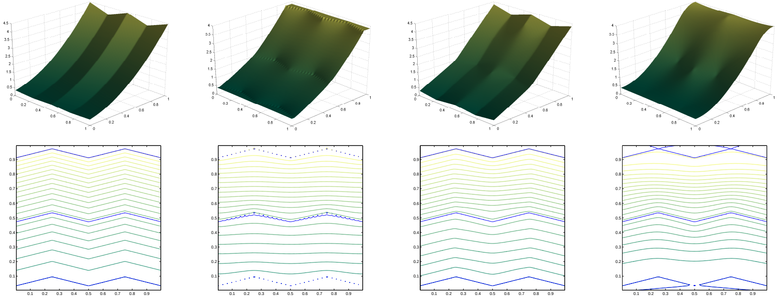


Related work

- Goal: sparse data, no explicit parameterization, at least continuous, sharp ridges, function space domain
- Related methods:
  - Explicit parameterization [Meyers et al. '92, Masnou/Morel '98, Hormann et al. '03, Meyer '11]
  - Geodesic distance transform [Soille '91]
  - Membrane, thin plate spline [Duchon '76] – smooth contours
  - Kriging [Matheron '71, Stein '99]
  - Anisotropic Diffusion [Desbrun '00], of normals [Tasdizen '02]
  - Total curvature [Elsey/Esedoglu '07]
  - Implicit representation [Ye et al. '10], higher-order TV [Lai/Tai/Chan '11]
  - Absolutely Minimizing Lipschitz Extension [Alvarez et al.'93, Caselles et al. '98] – $D^2u(Du/|Du|, Du/|Du|) = 0$,
  - many more...

Classic approaches

- Smoothed-out contours

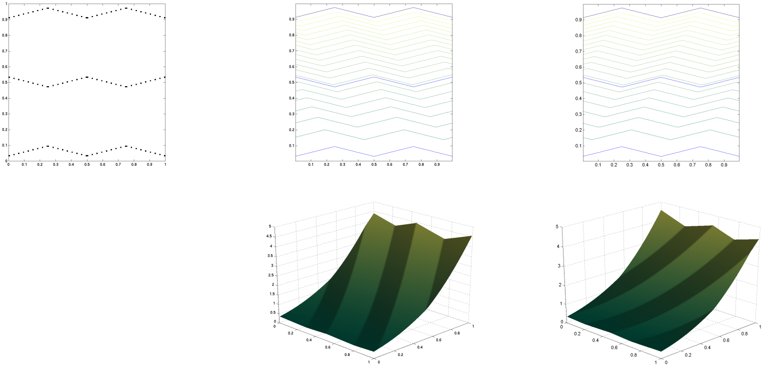


input | quadratic | TV^2 | TV^3

- convex?

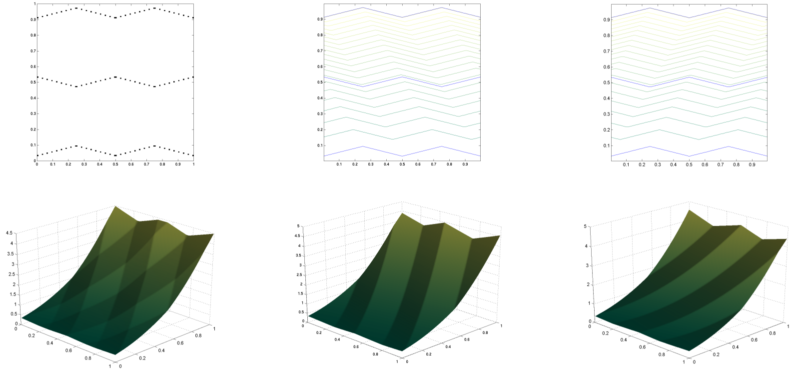
Challenges – 2

- Is convex regularization enough?



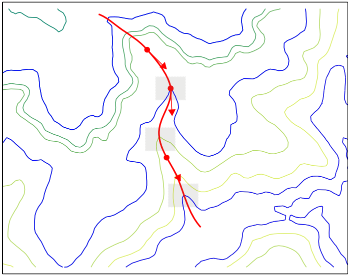
Challenges – 2

- Is convex regularization enough?



Auxiliary vectors

- Need to introduce non-convexity
- Contours are similar
- Smooth *across* level lines – vector field v associates points on contours



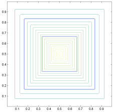
- Introduce anisotropy:

$$R_1^{(3)}(u) := \int_\Omega \|D^3 u(v,\cdot,\cdot)\|, R_2^{(3)}(u) := \int_\Omega \|D^3 u(v,v,\cdot)\|, R_3^{(3)}(u) := \int_\Omega \|D^3 u(v,v,v)\|,$$

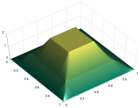
$$R_1^{(2)} := \int_\Omega \|D^2 u(v,\cdot)\|, R_2^{(2)} := \int_\Omega \|D^2 u(v,v)\|$$

Known vector field

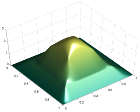
- Known v:



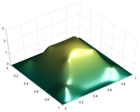
input



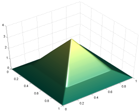
AMLE



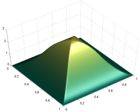
$\|D^2 u\|$



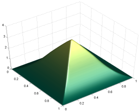
$\|D^3 u\|$



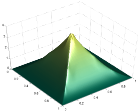
$\|D^2 u(v,\cdot)\|$, $|D^2 u(v,v)|$



$\|D^3 u(v,\cdot,\cdot)\|$



$\|D^3 u(v,v,\cdot)\|$

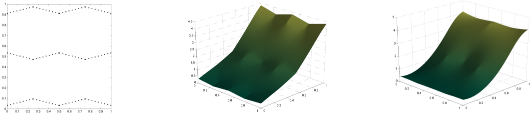


$|D^3 u(v,v,v)|$

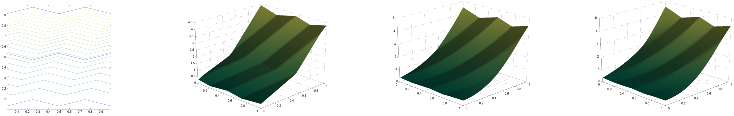
- $v = Du/|Du|$ is *not* enough (AMLE)

Ambiguities

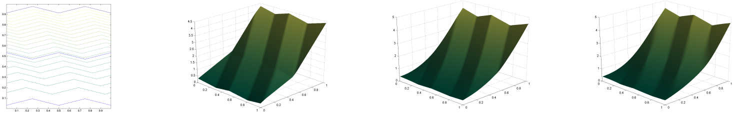
- Resolves ambiguities:



input $\|D^2 u\|$ $\|D^3 u\|$



$\|D^2 u(v,\cdot)\|$ $\|D^3 u(v,\cdot,\cdot)\|$ $\|D^3 u(v,v,\cdot)\|$



$\|D^2 u(v,\cdot)\|$ $\|D^3 u(v,\cdot,\cdot)\|$ $\|D^3 u(v,v,\cdot)\|$

Finding v

- *Idea:* v locally points into direction where *normal* to contour line changes least – assume contour lines are locally only translated:

$$v(x) \quad = \quad \arg \min_{w, \|w\|_2 = 1} \|K_\sigma * (D(Du/|Du|))(x)\, w\|_2.$$

- Enforce regularity where u is almost planar, decrease where v is accurate: normalize and solve

$$\min_{v'} \frac{1}{2} \int_\Omega w(x) \|v'(x) - v(x)\|_2^2 dx + \frac{\rho}{2} \int_\Omega \|Dv'(x)\|_2^2 dx.$$

where $w(x)$ is largest singular value of $K_\sigma * (D(Du/|Du|))(x)$.

Finding v

- Subtlety:

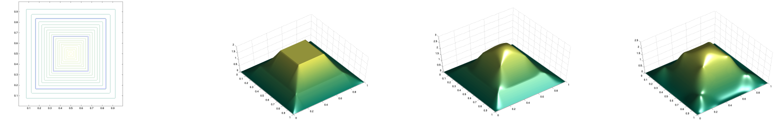
$$v(x) \quad = \quad \arg \min_{w, \|w\|_2 = 1} \|K_\sigma * (D(Du/|Du|))(x)\, w\|_2.$$

$$\min_{v'} \frac{1}{2} \int_\Omega w(x) \|v'(x) - v(x)\|_2^2 dx + \frac{\rho}{2} \int_\Omega \|Dv'(x)\|_2^2 dx.$$

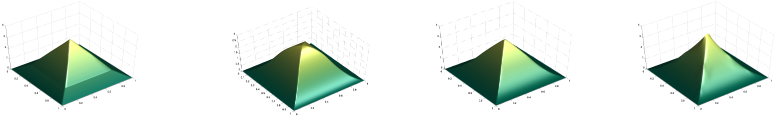
- Sign of v undefined, matters in second step. Normalize v so that $\langle v(x), Du(x) \rangle \leqslant 0$ (towards negative gradient),
- For unknown u, start with a random field v^0 and alternate between computing u^k and v^k
- Random v approximates isotropic regularizers $R_0^{(3)}$ or $R_0^{(2)}$, but has the additional advantage of introducing randomness that can solve ambiguous situations

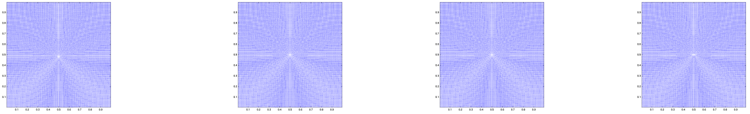
Adaptive choice of v

- We still obtain sharp edges with adaptive v:



original   AMLE $\|D^2 u\|$ $\|D^3 u\|$

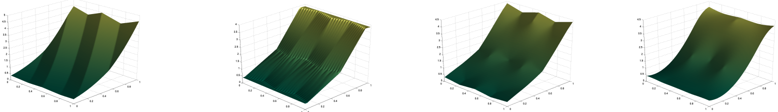




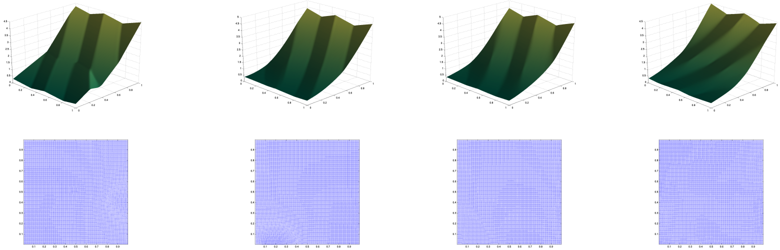
$\|D^2 u(v,\cdot)\|$, $|D^2 u(v,v)|$ $\|D^3 u(v,\cdot,\cdot)\|$ $\|D^3 u(v,v,\cdot)\|$ $|D^3 u(v,v,v)|$

Adaptive choice of v

- Still sharp edges, ambiguities are correctly resolved



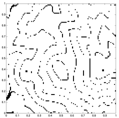
original AMLE $\|D^2 u\|$ $\|D^3 u\|$



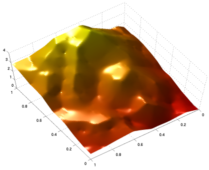
$\|D^2 u(v,\cdot)\|$, $|D^2 u(v,v)|$ $\|D^3 u(v,\cdot,\cdot)\|$ $\|D^3 u(v,v,\cdot)\|$ $|D^3 u(v,v,v)|$

Real-world results

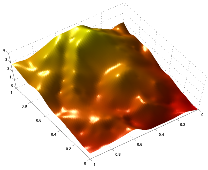
- "Bull mountain" from National Elevation Dataset [Gesch et al. '09]



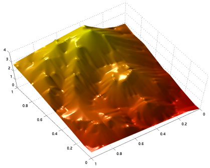
input



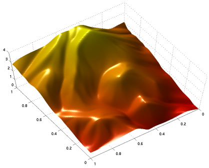
$\|D^2 u\|$



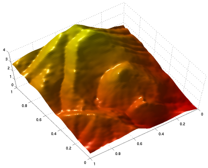
$\|D^3 u\|$



AMLE

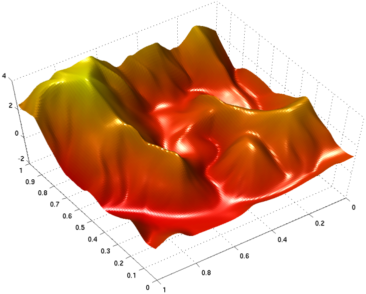


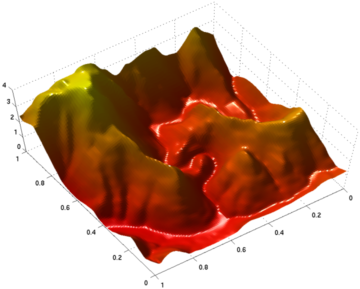
$\|D^3 u(v, \cdot, \cdot)\|$



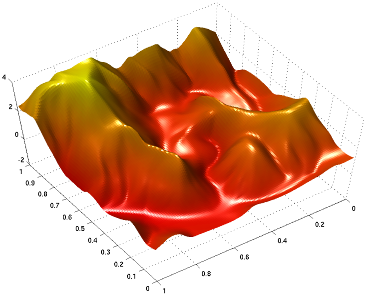
ground truth

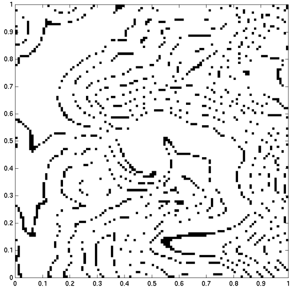
Real-world results





Real-world results





Real-world performance

- L^2 distance

#	quadratic	AMLE	$TV^{(3)}$	$R_1^{(3)}$	$R_2^{(3)}$	$R_3^{(3)}$	$TV^{(2)}$	$R_1^{(2)}$	$R_2^{(2)}$
1	141.57	138.15	15.10	**8.45**	10.99	14.46	30.59	18.80	25.64
2	95.55	83.91	13.61	**11.10**	13.41	21.28	25.29	19.18	28.74
3	202.06	235.86	29.61	**28.75**	33.42	50.23	84.71	43.34	97.27
4	79.33	56.36	18.93	**8.59**	10.34	13.74	26.04	13.35	21.41
5	103.76	88.91	34.47	**17.03**	21.96	26.23	43.84	23.10	28.53

- L^2 distance of normals

#	quadratic	AMLE	$TV^{(3)}$	$R_1^{(3)}$	$R_2^{(3)}$	$R_3^{(3)}$	$TV^{(2)}$	$R_1^{(2)}$	$R_2^{(2)}$
1	10.88	15.33	3.46	**2.12**	2.43	2.95	4.32	3.35	5.49
2	21.73	22.94	6.66	**5.77**	7.04	10.85	9.45	9.07	13.71
3	29.96	42.91	10.67	**10.49**	12.64	19.12	15.32	15.02	21.74
4	13.28	11.39	5.34	**3.31**	3.74	4.76	6.78	4.44	6.63
5	9.86	10.36	5.09	**3.15**	3.74	4.65	5.77	4.41	5.68

Summary

- **Non-convex variational surface interpolation**
  - Interesting problem, highlights properties of regularizers
  - Robust method to find surface and association between level lines