Adaptive Second-Order Total Variation: An Approach Aware of Slope Discontinuities

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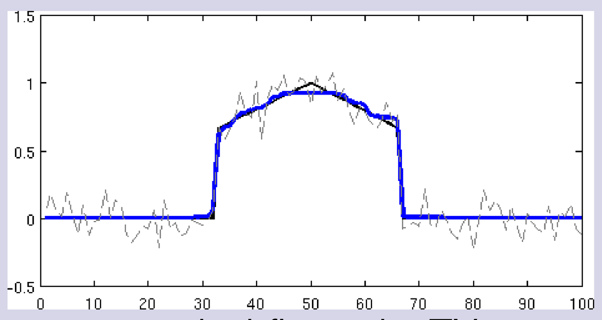
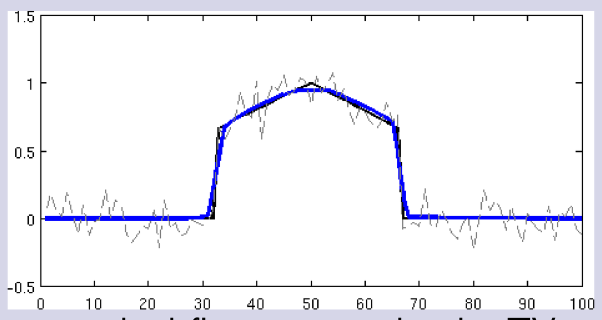
Overview

We propose an improved first- and second-order anisotropic total variation (TV) regularization, which in particular reduces smoothing across edges and slope discontinuities. The epigraph of the data is investigated with a special structure tensor to detect slope discontinuities.

Motivation

Example (1D):


standard first order TV


standard first + second order TV

- Standard TV suffers from stair-casing and flattening of the ridge.
- Second order TV avoids stair-casing, but smooths edges and ridge.

Detecting Edges

We follow the *standard approach* utilizing the structure tensor

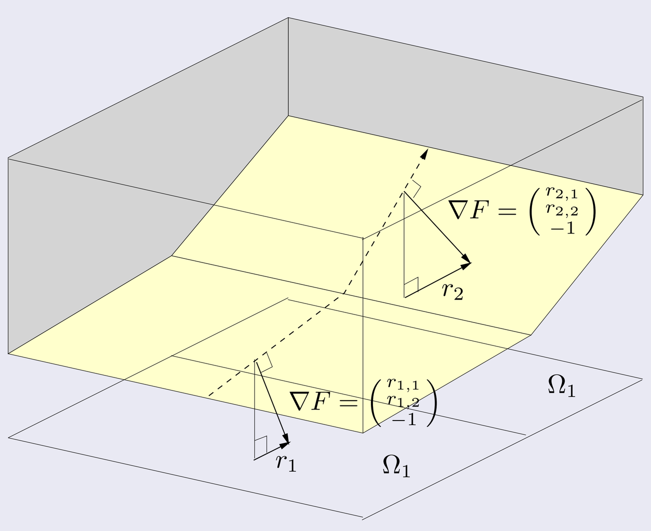
$$S_u(\mathbf{x}) = (\nabla u_\sigma(\mathbf{x}) \nabla u_\sigma(\mathbf{x})^\top)_\rho \qquad (1)$$

of image $u : \Omega \to \mathbb{R}$. $(\cdot)_\sigma$ and $(\cdot)_\rho$ denote the convolution with Gaussian kernels with std. deviation $\sigma, \rho > 0$.

- eigenvalues: $\lambda_1^S \geq \lambda_2^S \geq 0$
- eigenvectors v_1^S, v_2^S
- edge indicator function: $E^S(\mathbf{x}) := \min\{c \cdot (\lambda_1^S(\mathbf{x}) - \lambda_2^S(\mathbf{x})), 1\}, \quad c > 0$

Detecting Slope Discontinuities

Assumption: u is continuous and piecewise affine, with one discontinuity of ∇u along a line.



example of a slope discontinuity

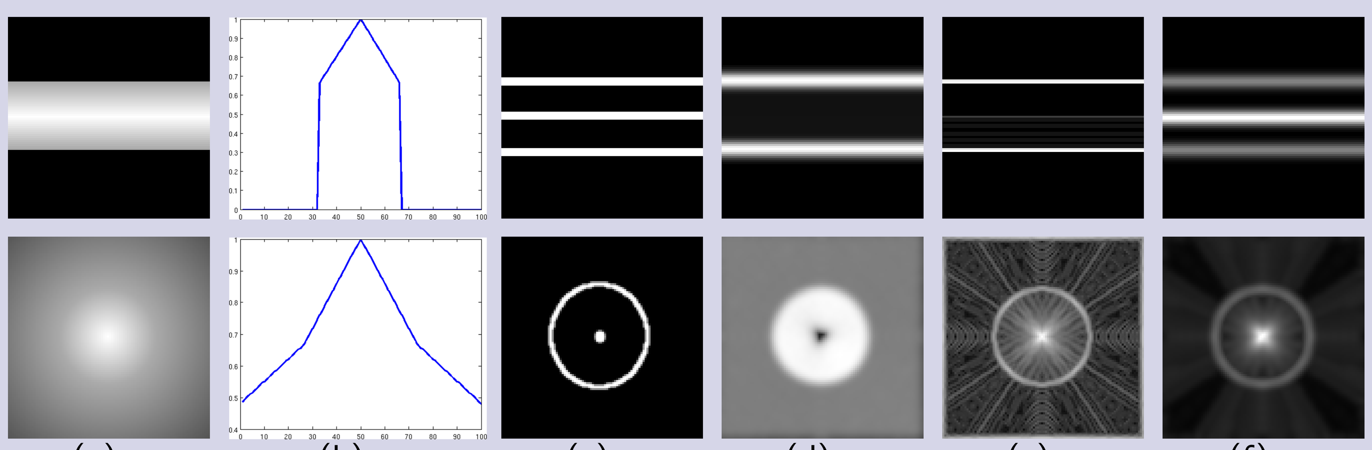
- graph of u: $\Gamma = (x, y, u(x,y))^\top$.
- epigraph of u: super-level set $\{(x,y,z) \mid F(x,y,z) \geq 0\}$ of $F(x,y,z) := u(x,y) - z$.
- to detect edges of the graph, we propose to apply the three-dimensional structure tensor to F:

$$\Big((\nabla F(x,y,z))(\nabla F(x,y,z))^\top \Big)_\rho . \qquad (2)$$

- restrict this structure tensor to Γ:

$$T_u(x,y) := \Big((\nabla \tilde{F}(x,y))(\nabla \tilde{F}(x,y))^\top \Big)_\rho , \qquad (3)$$

where $\nabla \tilde{F}(x,y) := \nabla F(x,y,u(x,y))$.
- eigenvalues: $\lambda_1^T \geq \lambda_2^T \geq \lambda_3^T \geq 0$.
- define indicator function for slope discontinuities: $E^T(\mathbf{x}) := \min\{c\lambda_2^T(\mathbf{x}), 1\}$



Detecting slope discontinuities: (a) test images, (b) cross-section, (c) ideal result, (d) standard structure tensor, (e) curvature based approach, (f) proposed method.

Approach: Anisotropic Second Order TV

Variational approach:

$$\mathcal{F}(u) := \frac{1}{2}\|u - f\|_{L^2}^2 + \mathcal{R}_1(u) + \mathcal{R}_2(u) \qquad u \in BV^2(\Omega) \qquad (4)$$

First order

$$\mathcal{R}_1(u) := \int_\Omega \Big(\nabla u^\top(\mathbf{x}) A(\mathbf{x}) \nabla u(\mathbf{x}) \Big)^{\frac{1}{2}} \, d\mathbf{x}, \qquad (5)$$

with

$$A(\mathbf{x}) = \Big(v_1^S(\mathbf{x}) \; v_2^S(\mathbf{x}) \Big) \begin{pmatrix} \alpha_1(\mathbf{x}) & 0 \\ 0 & \alpha_2(\mathbf{x}) \end{pmatrix} \begin{pmatrix} (v_1^S(\mathbf{x}))^\top \\ (v_2^S(\mathbf{x}))^\top \end{pmatrix}. \qquad (6)$$

Second order

For $u \in C^\infty(\Omega)$

$$\mathcal{R}_2(u) = \int_\Omega \beta_1 \|(\mathrm{H}u) v^S\|_2 + \beta_2 \|(\mathrm{H}u)(v^S)^\perp\|_2 \, dx, \qquad \text{(with H = Hessian of } u). \qquad (7)$$

Generalization to $u \in L^1(\Omega)$

$$\mathcal{R}_2(u) := \sup\Big\{ \int_\Omega (\operatorname{div}^2 \varphi) u \, dx \mid \varphi \in \mathcal{C} \Big\}, \text{ with} \qquad (8)$$

$$\mathcal{C} := \{ C_c^\infty(\Omega; \mathbb{R}^4), \forall \mathbf{x} \in \Omega : \langle \varphi(\mathbf{x}), w_1(\mathbf{x}) \rangle^2 + \langle \varphi(\mathbf{x}), w_2(\mathbf{x}) \rangle^2 \leq (\beta_1(\mathbf{x}))^2, \; \langle \varphi(\mathbf{x}), w_3(\mathbf{x}) \rangle^2 + \langle \varphi(\mathbf{x}), w_4(\mathbf{x}) \rangle^2 \leq (\beta_2(\mathbf{x}))^2 \}, \qquad (9)$$

with suitably chosen $w_i(\mathbf{x}) : \Omega \to \mathbb{R}^4$ depending on v^S.

Theorem: Assume that $\alpha_i(\mathbf{x}), \beta_i(\mathbf{x})$, $i = 1, 2$, are bounded from below and above by positive constants. Then there exists a unique minimizer of (4).

Proof: see SSVM paper.

Choice of $\alpha_i(\mathbf{x}), \beta_i(\mathbf{x})$:

Let

$$E(\mathbf{x}) := \max\{E^S(\mathbf{x}), E^T(\mathbf{x})\} \in [0, 1]. \qquad (10)$$

The indicator functions E^S, E^T are calculated from the structure tensors S_f, and T_f applied to data f.

$$\begin{aligned} \alpha_1(\mathbf{x}) &:= E(\mathbf{x})\alpha_{min} + (1 - E(\mathbf{x}))\alpha_{max}, & \alpha_2(\mathbf{x}) &:= \alpha_{max}, \\ \beta_1(\mathbf{x}) &:= E(\mathbf{x})\beta_{min} + (1 - E(\mathbf{x}))\beta_{max}, & \beta_2(\mathbf{x}) &:= \beta_{max}, \end{aligned} \qquad (11)$$

with suitable $\alpha_{min}, \alpha_{max}, \beta_{min}, \beta_{max} > 0$.

Related approaches: adaptive TGV / infimal convolution

Infimal convolution (IC) [Setzer et al. 2011]:

$$\min_u \frac{1}{2}\|u - f\|_{l^2}^2 + \inf_{\{u_1, u_2 | \nabla u = u_1 + u_2\}} \alpha \|u_1\|_{l^1} + \beta \|Du_2\|_{l^1}. \qquad (12)$$

TGV [Bredies et al. 2010]:

$$TGV^2(u) := \sup\{ u \operatorname{div}^2 \varphi \, dx \mid \varphi \in C_c^2(\Omega, Sym^2), |\varphi| \leq \alpha, |\operatorname{div} \varphi| \leq \beta \text{ a.e. } \}. \qquad (13)$$

Adaptive variants of IC and TGV:

$$\alpha(\mathbf{x}) := E(\mathbf{x})\alpha_{min} + (1 - E(\mathbf{x}))\alpha_{max}, \qquad \beta(\mathbf{x}) := E(\mathbf{x})\beta_{min} + (1 - E(\mathbf{x}))\beta_{max}, \qquad (14)$$

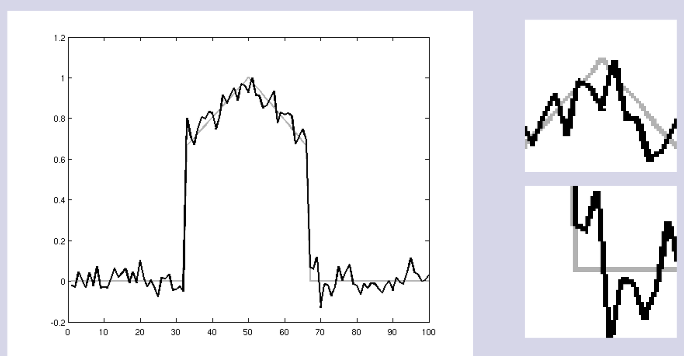
with suitable $\alpha_{min}, \alpha_{max}, \beta_{min}, \beta_{max}$ and E as defined in (10).

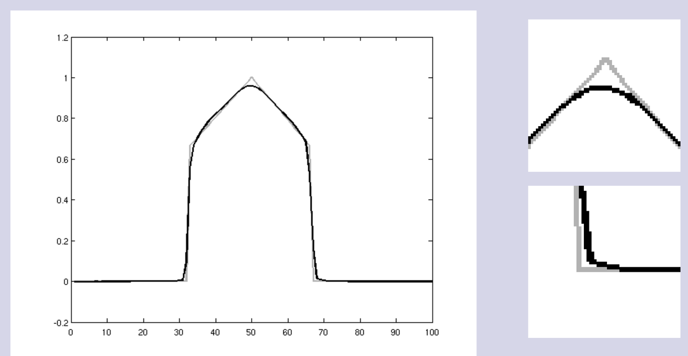
Results

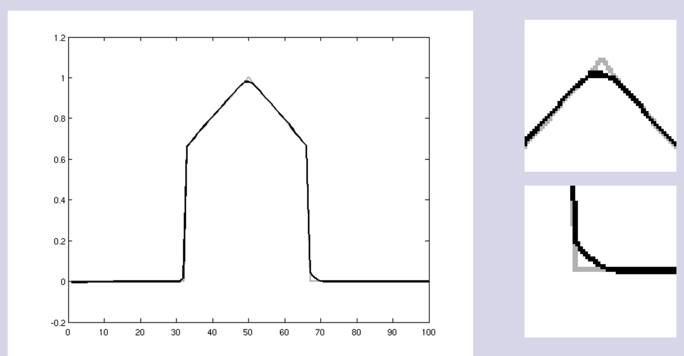
Methods to compare:

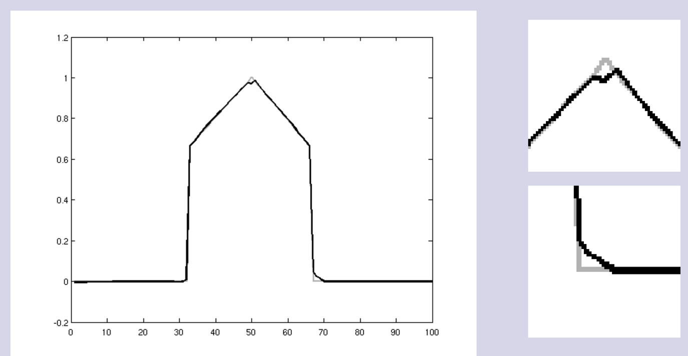
- TV^2 – standard second-order TV
- IC – infimal convolution (standard/adaptive)
- TGV (standard/adaptive)
- $aTV^2(E = E^S)$ – proposed approach using $E = E^S$
- aTV^2 – proposed approach (E as in (10)).

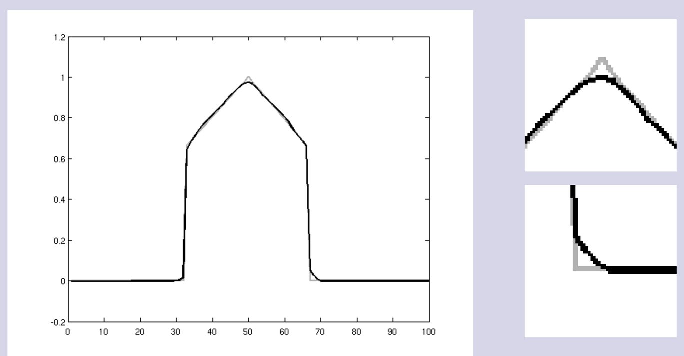
Qualitative comparison

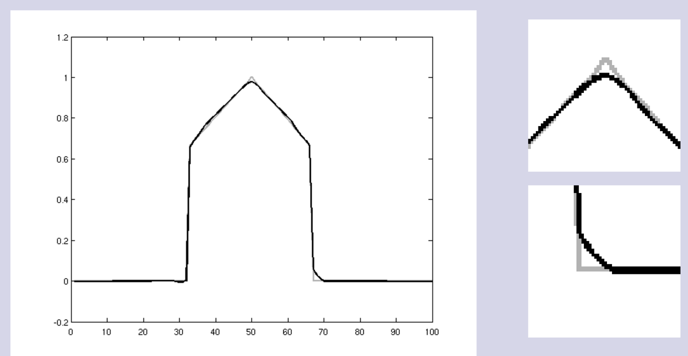

noisy image

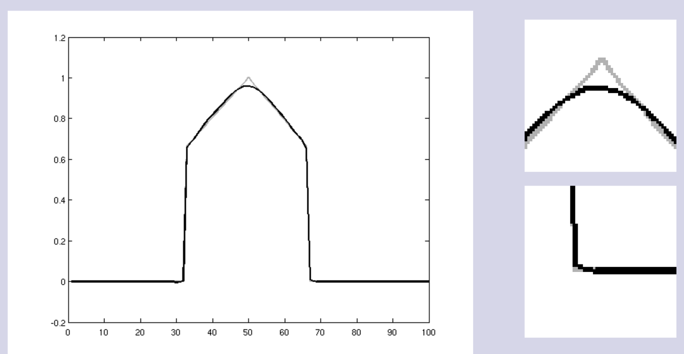

TV^2

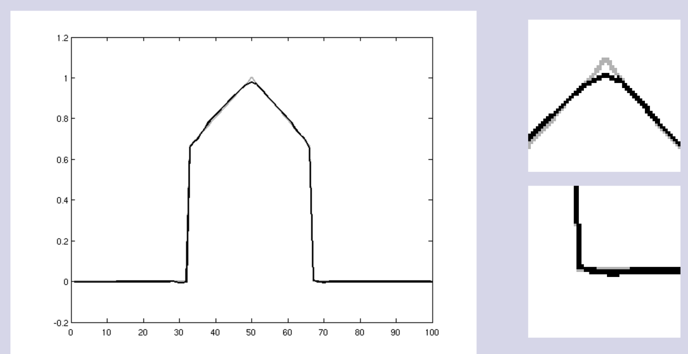

TGV


adaptive TGV


IC

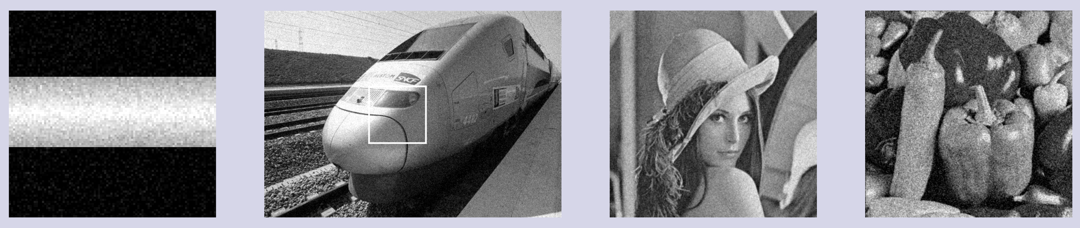

adaptive IC


aTV^2 with $E = E^S$


aTV^2

Cross-sections of the results (back lines) of TGV, IC, their adaptive variants and the proposed method. Noise-free data shown in gray.

Quantitative comparison


test images used for comparison.

Example	Roof	Train (part)	Lena	Peppers
aTV^2 $(E = E^S)$	4.6946e-4	2.4249e-4	1.0703e-4	1.8858e-4
TGV	0.8857e-4	2.3644e-4	0.9362e-4	1.3883e-4
Adaptive TGV	0.8703e-4	2.3364e-4	0.8985e-4	1.3258e-4
IC	1.0405e-4	2.3968e-4	0.9519e-4	1.3822e-4
Adaptive IC	0.9861e-4	2.3693e-4	0.9205e-4	1.3589e-4
aTV^2	**0.5703e-4**	**2.2560e-4**	**0.8749e-4**	**1.2997e-4**

Mean squared errors (MSE) to the noise-free image for the different methods. Approximate optimal parameters for each method were retrieved by grid search.

Bibliography

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