Dynamical Optimal Transport and Functional Lifting

Thomas Vogt, Roland Haase, Danielle Bednarski, Frederic Kanter,
Willem Diepeveen, Caterina Rust, and Jan Lellmann, University of Lübeck

Mathematics and Image Analysis
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MIC Institute of Mathematics and Image Computing

UNIVERSITÄT ZU LÜBECK

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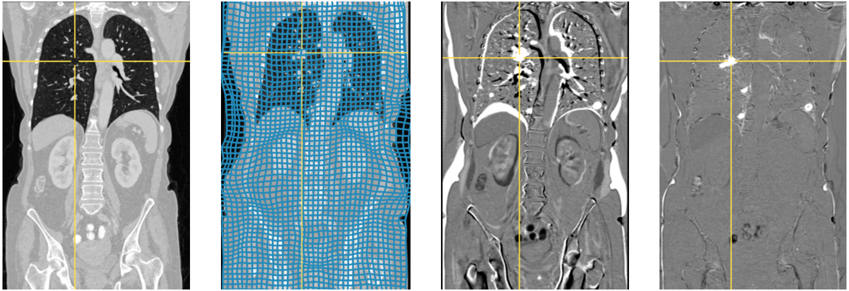
Motivation – Motion Estimation



I^0 – CT before | I^1 – CT after | $I^1 - I^0$ without registration | with registration

Goal: find “good” deformation $u : \Omega \to \mathbb{R}^2$ that maps points from I^0 to I^1: $x \mapsto x + u(x)$.

Images: *König, Rühaak, Derksen, Lellmann’18*

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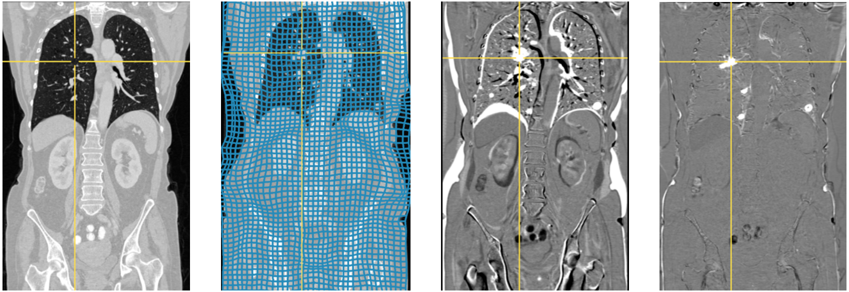

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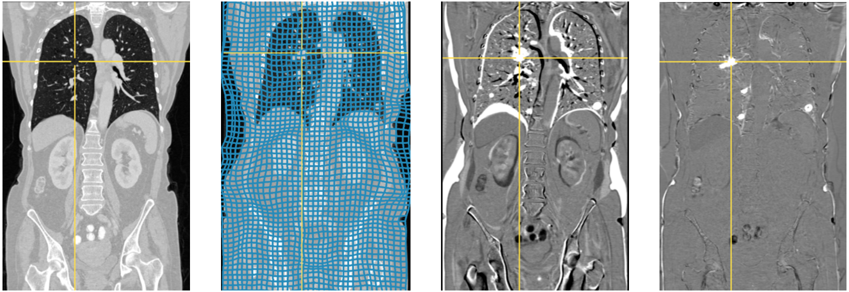
$$\inf_{u:\Omega\to\mathbb{R}^2} \int_\Omega |I^0(x) - I^1(x + u(x))|^2 \, dx + \lambda \int_\Omega \|\nabla^2 u(x)\|^2 \, dx$$

Images: *König, Rühaak, Derksen, Lellmann’18*

Motivation – Motion Estimation



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$$\inf_{u:\Omega\to\mathbb{R}^2} \int_\Omega g(x, u(x), \nabla^2 u(x))\, dx$$

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Motivation – Motion Estimation

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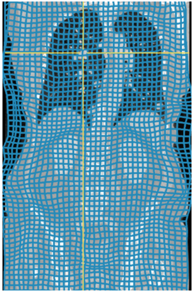
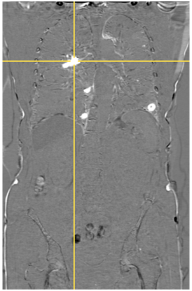


I^1 – CT after

$I^1 - I^0$ without registration



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$$\inf_{u:\Omega\to\mathbb{R}^2} \int_\Omega g(x, u(x), \nabla^2 u(x))\, dx$$

highly non-convex energy (but convex in $\nabla^2 u$)

Images: *König, Rühaak, Derksen, Lellmann'18*

Imaging Live

IMAGING LIVE
Mathematics in action

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Live

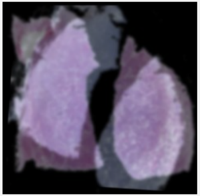


Image registration is the task of finding a transformation that best aligns two given images. A typical problem could be that we have two images of a patient taken at different points in time, and the patient has moved in between.

This page demonstrates the use of *rigid* image registration, where one assumes that the images only have to be translated and rotated in order to align them.

Drag with the mouse or finger to move the template image. Rotate using two fingers. The algorithm will automatically try to restore the alignment.

Simple algorithms such as this one have difficulties with complicated objects: they fill only find a *local minimum*, which is only partially optimal, and from which the algorithm cannot escape.

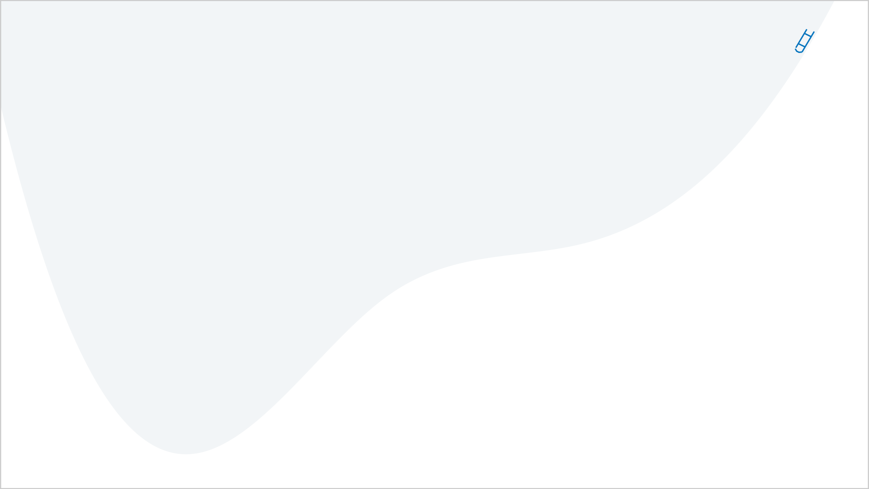
Authors: Lars Koenig, Jan Lellmann

Show demo

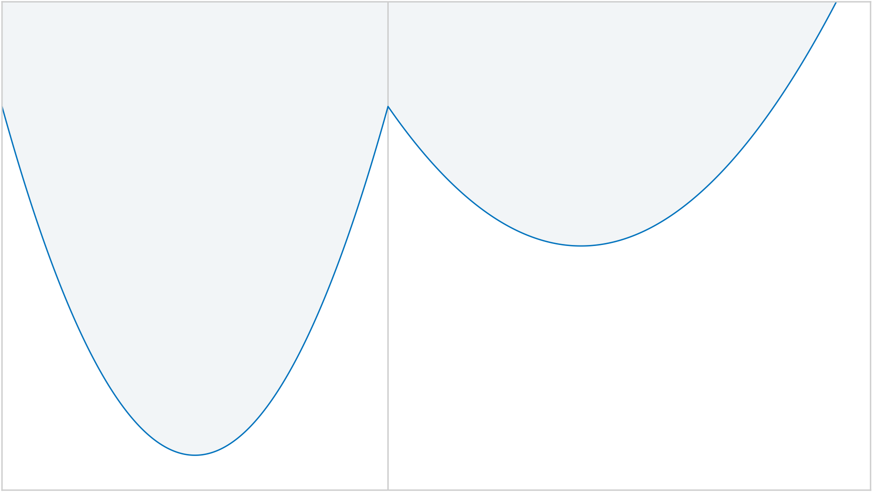
Aside

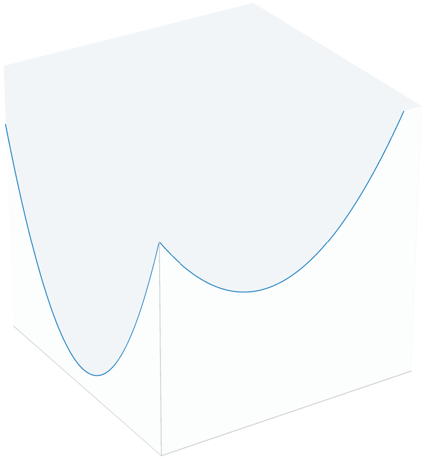
https://imaging.live

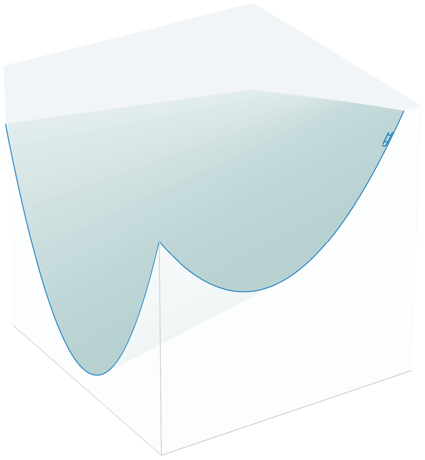
How to find a good minimizer?

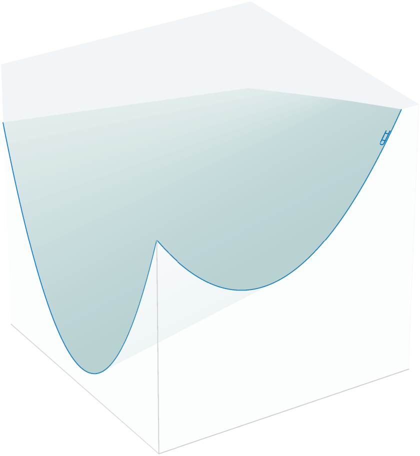


?



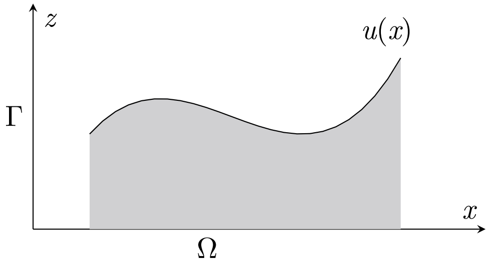


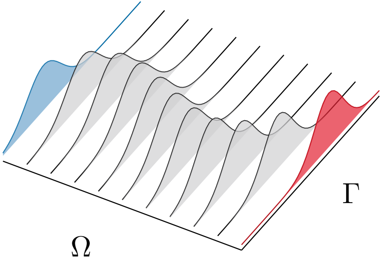




$$\min_{u\in\mathcal{U}} f(u) \quad \to \quad \min_{u'\in\mathcal{U}'\supseteq\mathcal{U}} f'(u')$$

Lifting





A map $\Omega \to \Gamma$
A curve in $\Omega \times \Gamma$ (gph u)

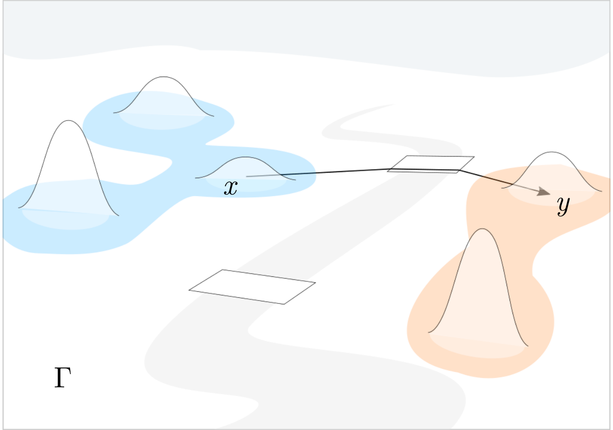
A measure in $\Omega \times \Gamma$
A measure-valued map $u' : \Omega \to \mathcal{P}(\Gamma)$

$$u(x) = c \quad \leftrightarrow \quad u'(x) = \delta_c$$

$$\min_{u:\Omega\to\Gamma} f(u) \quad \to \quad \min_{u':\Omega\to\mathcal{P}(\Gamma)} f'(u')$$

Tricky: How to extend f? Size? Convexity? Artificial minimizers?

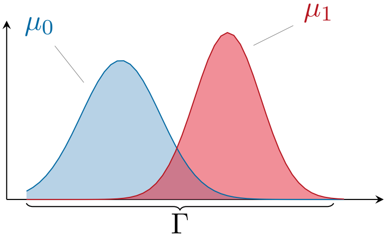
Optimal Transport



Transporting one unit from x to y costs $c(x, y)$. Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\Gamma)$, find transport plan $\gamma \in \mathcal{P}(\Gamma \times \Gamma)$ with minimal total cost.

Image: M. Hänel

Wasserstein Distance



Kantorovich's Problem of Optimal Transport

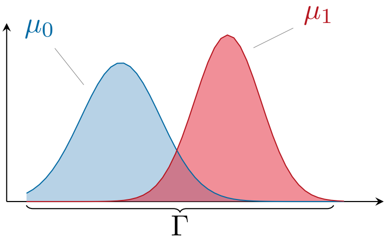
Given measures $\mu_0, \mu_1 \in \mathcal{P}(\Gamma)$,

$$\text{minimize} \int_{\Gamma\times\Gamma} c(x,y)\, d\gamma(x,y) \quad \text{s.t.} \quad \pi_x\gamma = \mu_0,\ \pi_y\gamma = \mu_1.$$

Infimum for $c(x,y) = \frac{1}{p} d(x,y)^p$ is the p-Wasserstein distance $W_p(\mu_0,\mu_1)^p$, $1 \leq p < \infty$

Monge problem: Monge 1781; KR: Kantorovich'42; Kantorovich, Rubinstein'57

Wasserstein Distance



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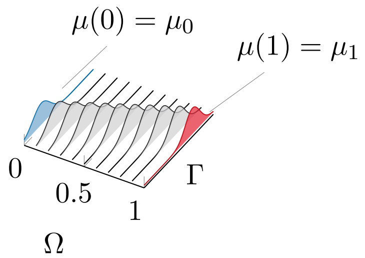
Monge problem: Monge 1781; KR: Kantorovich'42; Kantorovich, Rubinstein'57; Registration: Haker et al.'04; Segmentation, shape matching: Chan, Esedoglu'09; Swoboda, Schnörr'13; Schmitzer, Schnörr'13; Image and shape retrieval: Rubner, Tomasi, Guibas'00–02; Rabin, Peyré'10; Color transfer: Rabin, Peyré'11; Displacement interpolation, gradient flows: McCann'97; Benamou, Brenier'00; Burger, Carillo, Wolfram'10; Düring, Mattes, Milisic'10; Papadakis, Peyré, Oudet'14; Books: Hewitt, Stromberg'65; Rachev, Rüschendorf'98; Evans, Gangbo'99; Weaver'99; Ambrosio'03; Villani'09; Santambrogio'15; Peyré'19

Dynamical Optimal Transport



Benamou-Brenier'00

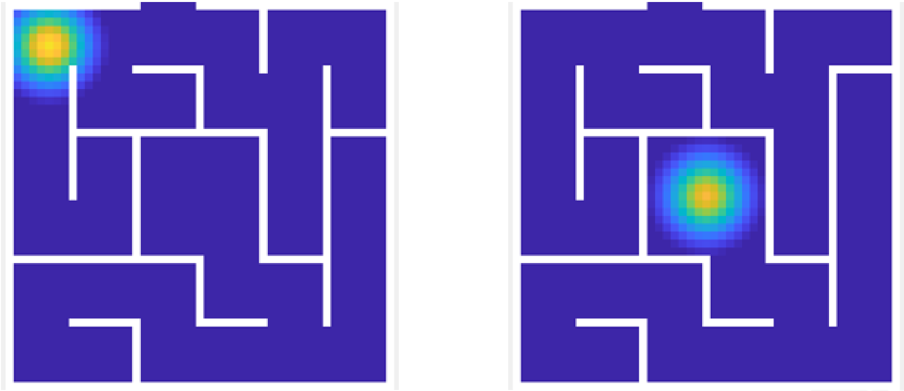
Now: $\rho(z, t)$. Assume the μ_i have densities, $\mu_i = \rho_i \mathcal{L}^d$. Then

$$W_2(\mu_0, \mu_1) = \inf\left\{ \int_0^1 \int_\Gamma \frac{1}{2} \frac{\|m(z,t)\|_2^2}{\rho(z,t)}\, dz\, dt : \partial_t \rho + \operatorname{div}_z m = 0, \right.$$
$$\left. \rho(\cdot, 0) = \rho_0, \rho(\cdot, 1) = \rho_1 \right\}$$

Continuity equation links density $\rho \leftrightarrow$ momentum m (= mass $\cdot$ velocity). Convex in ρ, m!

DOT: Benamou, Brenier'00; Extended MKT: Brenier'03; Brasco'10; Santambrogio'15; General Integrands: Jiminez'08

Dynamical Optimal Transport



Inspired by: Papadakis, Peyré, Oudet'14

...back to Image Processing

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Functional Lifting

Non-convex Variational Problem

$$\inf_{u \in \mathcal{U}} \underbrace{\int_\Omega g(x, u(x), Lu(x)) \,\mathrm{d}x}_{=: f(u)}$$

$\Omega \subseteq \mathbb{R}^d$	open image domain
$\Gamma \subseteq \mathbb{R}^s$	compact range
L	linear operator $(\nabla, \nabla^2, \Delta)$
$g : \Omega \times \Gamma \times \mathbb{R}^{ds} \to \mathbb{R}$ convex only in p	

This formulation: *Vogt, Haase, Bednarski, Lellmann'20*; Second-order lifting: Löwenhauser, Lellmann'18; Lellmann, Vogt'19

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$$\underbrace{\inf_{(u,p)\in\mathcal{V}} \int_\Omega g(x, u(x), p(x))\,\mathrm{d}x}_{=:\tilde{f}(u,p)} \quad \text{s.t.} \quad p = Lu$$

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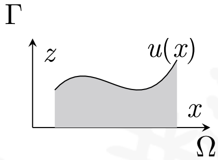
Strategy

- Replace (u, p) by measures (μ, E) on $\Omega \times \Gamma$ concentrated on $\operatorname{gph} u$ – “lifting”: $u \leftrightarrow \delta_u$, $p \leftrightarrow p\delta_u$
- Replace $\tilde{f}$ by convex functional $f : \mathcal{M}(\Omega \times \Gamma)^{1+ds} \to \mathbb{R}$



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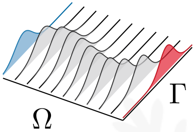

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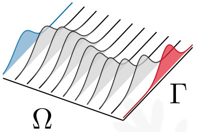
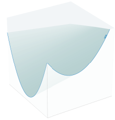


Components

1. Lift f: If $\mu = \delta_u$ and $E = p\delta_u$, then $f(E, \mu) \overset{!}{=} \tilde{f}(u, p)$
2. Translate equation $p = Lu$ to equation between E and μ



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Theorem Lifting

Generalized Benamou-Brenier [Vogt, Haase, Bednarski, Lellmann'20, Def. 1]

For $V \subset \mathbb{R}^n$ *(here: $V = \Omega \times \Gamma$)*, let $g : V \times \mathbb{R}^{ds} \to [0, \infty]$ be convex in the second variable and lower semi-continuous *(here: $g((x, u), p)$)*. Then the generalized Benamou-Brenier functional $\mathcal{B}_g : \mathcal{M}(V)^{ds+1} \to [-\infty, \infty] =: \overline{\mathbb{R}}$ is defined as

$$\mathcal{B}_g(\nu) := \sup_{\phi \in \mathcal{K}_g} \langle \nu, \phi \rangle, \quad \nu = (\mu, E),$$

where

$$\mathcal{K}_g := \{(\phi^\xi, \phi^\lambda) \in C_0(V, \mathbb{R}^{ds} \times \mathbb{R}) : g^*(t, \phi^\xi(t)) + \phi^\lambda(t) \leq 0 \, \forall t \in V\}$$

is the set of dually admissible vector fields.

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Consequence [Vogt, Haase, Bednarski, Lellmann'00, Cor. 10]

$$(\mu, E) = (\delta_u, p \cdot \delta_u) \quad \Longrightarrow \quad \mathcal{B}_g((E, \mu)) = \int_\Omega g(x, u, p)\, dx.$$

Complete model

Lifting with Dynamical Optimal Transport

$$\inf_{\substack{\mu \in L^\infty_w(\Omega, \mathcal{P}(\Gamma)) \subset \mathcal{M}(\Omega \times \Gamma) \\ E \in \mathcal{M}(\Omega \times \Gamma)^{ds}}} \left\{ \mathcal{B}_g(E, \mu) : \nabla_x \mu + \operatorname{div}_z E = 0 \right\}$$

Vogt, Haase, Bednarski, Lellmann'20; Benamou, Brenier'00; Calibrations: Brakke'91; Alberti et al.'03; Chan et al.'06; Pock et al.'08/10/19; Bouchitté et al.'18; Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Currents: Möllenhoff, Cremers'19; First-order: Brenier'18; Ghoussub et al.'20; Roto-Translation/Curvature: Bredies, Pock'15; Chambolle, Pock'19; Böttcher, Wirth'20

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Constraint: generalized continuity equation for $g(x, u, \nabla u)$.

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Compare Benamou-Brenier (DOT)

$$\inf_{\rho, m} \left\{ \int_{[0,1] \times \Gamma} \frac{1}{2} \frac{\|m(z,t)\|_2^2}{\rho(z,t)} \, dz \, dt : \partial_t \rho + \operatorname{div}_z m = 0, \right.$$
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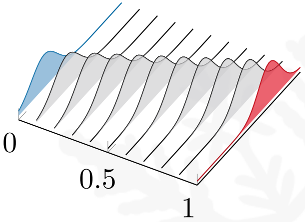

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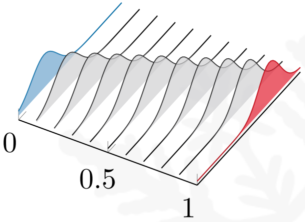

$\rightarrow$ special case $\Omega = [0, 1]$, $g(x, u, \nabla u) = \frac{1}{2}\|\nabla u(x)\|^2$.

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Complete model

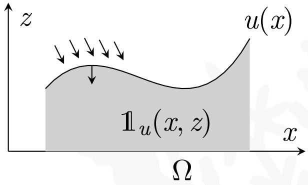
Lifting with Dynamical Optimal Transport

$$\inf_{\substack{\mu \in L^\infty_w(\Omega, \mathcal{P}(\Gamma)) \subset \mathcal{M}(\Omega \times \Gamma) \\ E \in \mathcal{M}(\Omega \times \Gamma)^{ds}}} \left\{ \mathcal{B}_g(E, \mu) : \nabla_x \mu + \operatorname{div}_z E = 0 \right\}$$

Constraint: generalized continuity equation for $g(x, u, \nabla u)$.

Compare Calibration-based Lifting

$$\inf_{v \in \mathrm{BV}(\Omega \times \mathbb{R})} \sup_{\phi \in \mathcal{K}'_f} \int_{\Omega \times \mathbb{R}} \langle \phi, Dv \rangle$$



Vogt, Haase, Bednarski, Lellmann'20; Benamou, Brenier'00; Calibrations: Brakke'91; Alberti et al.'03; Chan et al.'06; Pock et al.'08/10/19; Bouchitté et al.'18; Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Currents: Möllenhoff, Cremers'19; First-order: Brenier'18; Ghoussub et al.'20; Roto-Translation/Curvature: Bredies, Pock'15; Chambolle, Pock'19; Böttcher, Wirth'20

Complete model

Lifting with Dynamical Optimal Transport

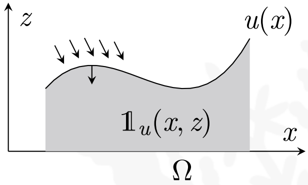
$$\inf_{\substack{\mu \in L^\infty_w(\Omega, \mathcal{P}(\Gamma)) \subset \mathcal{M}(\Omega \times \Gamma) \\ E \in \mathcal{M}(\Omega \times \Gamma)^{ds}}} \left\{ \mathcal{B}_g(E, \mu) : \nabla_x \mu + \operatorname{div}_z E = 0 \right\}$$

Constraint: generalized continuity equation for $g(x, u, \nabla u)$.

Compare Calibration-based Lifting

$$\inf_{v \in \mathrm{BV}(\Omega \times \mathbb{R})} \mathcal{B}_f(D_x v, -D_z v)$$

$\rightarrow$ special case $\Gamma \subseteq \mathbb{R}$, enforces regularity by $(E, \mu) = Dv$.



Vogt, Haase, Bednarski, Lellmann'20; Benamou, Brenier'00; Calibrations: Brakke'91; Alberti et al.'03; Chan et al.'06; Pock et al.'08/10/19; Bouchitté et al.'18; Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Currents: Möllenhoff, Cremers'19; First-order: Brenier'18; Ghoussub et al.'20; Roto-Translation/Curvature: Bredies, Pock'15; Chambolle, Pock'19; Böttcher, Wirth'20

Complete model

Lifting with Dynamical Optimal Transport

$$\inf_{\substack{\mu\in L^\infty_w(\Omega,\mathcal{P}(\Gamma))\subset\mathcal{M}(\Omega\times\Gamma)\\ E\in\mathcal{M}(\Omega\times\Gamma)^{ds}}} \left\{ \mathcal{B}_g(E,\mu) : \nabla_x \mu + \operatorname{div}_z E = 0 \right\}$$

Constraint: generalized continuity equation for $g(x, u, \nabla u)$.

New model: non-scalar domain, non-scalar range, & higher-order operators

Vogt, Haase, Bednarski, Lellmann'20; Benamou, Brenier'00; Calibrations: Brakke'91; Alberti et al.'03; Chan et al.'06; Pock et al.'08/10/19; Bouchitté et al.'18; Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Currents: Möllenhoff, Cremers'19; First-order: Brenier'18; Ghoussub et al.'20; Roto-Translation/Curvature: Bredies, Pock'15; Chambolle, Pock'19; Böttcher, Wirth'20

Complete model

Lifting with Dynamical Optimal Transport

$$\inf_{\substack{\mu \in L^\infty_w(\Omega, \mathcal{P}(\Gamma)) \subset \mathcal{M}(\Omega \times \Gamma) \\ E \in \mathcal{M}(\Omega \times \Gamma)^{ds}}} \left\{ \mathcal{B}_g(E, \mu) : -\nabla_x^2 \mu - \operatorname{div}_z E + \operatorname{div}_z^2 H = 0 \right\}$$

Constraint: generalized continuity equation for $g(x, u, \nabla^2 u)$.

New model: non-scalar domain, non-scalar range, & higher-order operators

Vogt, Haase, Bednarski, Lellmann'20; Benamou, Brenier'00; Calibrations: Brakke'91; Alberti et al.'03; Chan et al.'06; Pock et al.'08/10/19; Bouchitté et al.'18; Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Currents: Möllenhoff, Cremers'19; First-order: Brenier'18; Ghoussub et al.'20; Roto-Translation/Curvature: Bredies, Pock'15; Chambolle, Pock'19; Böttcher, Wirth'20

Complete model

Lifting with Dynamical Optimal Transport

$$\inf_{\substack{\mu \in L^\infty_w(\Omega, \mathcal{P}(\Gamma)) \subset \mathcal{M}(\Omega \times \Gamma) \\ E \in \mathcal{M}(\Omega \times \Gamma)^{ds}}} \left\{ \mathcal{B}_g(E, \mu) : -\Delta_x \mu - \operatorname{div}_z E + \operatorname{div}_z^2 H = 0 \right\}$$

Constraint: generalized continuity equation for $g(x, u, \Delta u)$.

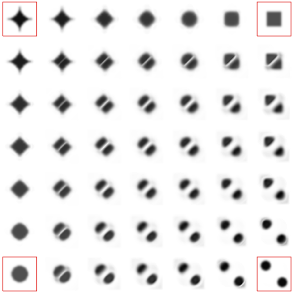
New model: non-scalar domain, non-scalar range, & higher-order operators

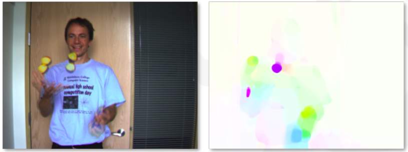
Vogt, Haase, Bednarski, Lellmann'20; Benamou, Brenier'00; Calibrations: Brakke'91; Alberti et al.'03; Chan et al.'06; Pock et al.'08/10/19; Bouchitté et al.'18; Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Currents: Möllenhoff, Cremers'19; First-order: Brenier'18; Ghoussub et al.'20; Roto-Translation/Curvature: Bredies, Pock'15; Chambolle, Pock'19; Böttcher, Wirth'20

Applications

I – Nonconvexity

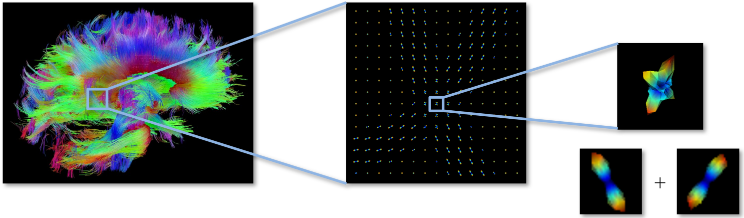
- **Connection to dynamical OT:** On the Connection between Dynamical Optimal Transport and Functional Lifting. T. Vogt, R. Haase, D. Bednarski, J. Lellmann, Preprint 2020.
- **Higher-order regularization:** Functional Liftings of Vectorial Variational Problems with Laplacian Regularization. T. Vogt, J. Lellmann, SSVM'19.
- Functional Lifting for Variational Problems with Higher-Order Regularization. B. Loewenhauser, J. Lellmann, IVLOPDE'18.
- **Motion estimation:** Sublabel-Accurate Convex Relaxation of Vectorial Multilabel Energies E. Laude, T. Möllenhoff, M. Möller, J. Lellmann, D. Cremers. ECCV'16.
- **Efficient discretization:** Sublabel–Accurate Relaxation of Nonconvex Energies. T. Möllenhoff, E. Laude, M. Möller, J. Lellmann, D. Cremers. Best Paper Honorable Mention at CVPR'16.



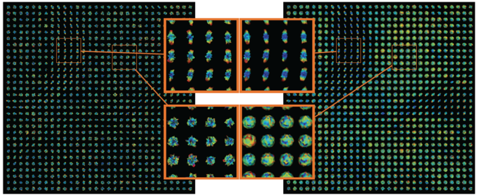


First-order: Pock, Cremers, Bischof, Chambolle'10; Higher-order: Ranftl, Pock, Bischof'13; Strecke, Goldlücke'18; Currents: Möllenhoff, Cremers'19; Harmonic mappings: Brenier'03; Lavenant'19; Interpolation: Solomon et al.'15

II – Naturally Measure-Valued



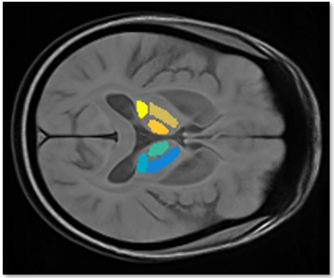
- **TV for measure spaces:** Measure-Valued Variational Models with Applications to Diffusion-Weighted Imaging. T. Vogt, J. Lellmann, Journal of Mathematical Imaging and Vision 2018.
- **DWI:** An Optimal Transport-Based Restoration Method for Q-Ball Imaging. T. Vogt, J. Lellmann, Scale Space Var. Meth. Comp. Vis. (SSVM), 2017

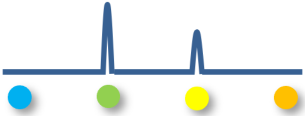


Dataset: Human Connectome Project

III – Segmentation/Labeling

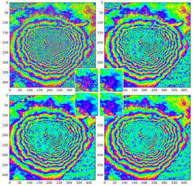
- **Overview:** Discrete and Continuous Models for Partitioning Problems. J. Lellmann, B. Lellmann, F. Widmann, and C. Schnörr, International Journal of Computer Vision, 2013.
- **Bounds:** Optimality Bounds for a Variational Relaxation of the Image Partitioning Problem. J. Lellmann, F. Lenzen, and C. Schnörr, Journal of Math. Imaging and Vision, 2012
- **Application to MRI:** A multi-contrast MRI approach to thalamus segmentation V. Corona. J. Lellmann, P. Nestor, C.-B. Schönlieb, J. Acosta-Cabronero, Human Brain Mapping, 2020.
- **Embeddings:** Continuous Multiclass Labeling Approaches and Algorithms. J. Lellmann and C. Schnörr, SIAM Journal on Imaging Sciences, 2011.
- **Multiclass:** Convex Optimization for Multi-Class Image Labeling with a Novel Family of Total Variation Based Regularizers. J. Lellmann, F. Becker, and C. Schnörr, ICCV'09



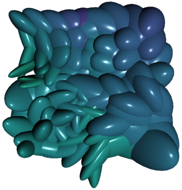


Finite domain & range: Potts'52; Boykov et al.'98/'01; Kleinberg, Tardos'01; Ishikawa'03; Function space: Chambolle, Cremers, Pock'12; Bae, Yuan, Tai'10/'11; Strang'83; Chan, Esedoglu, Nikolova'06; Zach et al.'08; Continuous min-cut/max-flow: Yuan, Bae, Tai, Boykov'10-11; Boyd, Bae, Tai, Bertozzi'18; Bounds: Strang'83; Chan, Esedoglu, Nikolova'06; Zach et al.'09; Olsson et al.'09

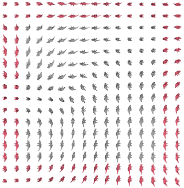
IV – Manifolds



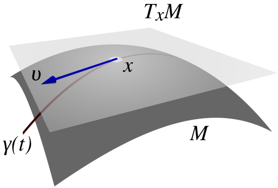
InSAR
$\mathcal{S}(1)$



DT-MRI
$\mathcal{P}(3)$



Orientations
$SO(3)$



- **Regularization:** Total Variation Regularization for Functions with Values in a Manifold. J. Lellmann, E. Strekalovskiy, S. Koetter, and D. Cremers, ICCV'13.
- **Discretization & higher-order:** Lifting methods for manifold-valued variational problems. T. Vogt, J. Lellmann, Handbook of Variational Methods for Nonlinear Geometric Data, 2020.

Problem

Find the minimizer of

$$\inf_{u:\Omega\to\mathcal{M}} f(u)$$

$\mathcal{M}$ Riemannian manifold

InSAR: Rocca, Prati, Ferreti'97; SO(3) interpolation: Vogt, Strekalovskiy, Cremers, Lellmann'20; Absil, Gousenbourger, Striewski, Wirth'16; Gousenbourger, Massart, Absil'19; $\mathcal{P}(3)$: Diepeveen'20; Denoising overview: Bergmann, Laus, Persch, Steidl'18

Take-home

- Generalized Benamou-Brenier = Lifting for functions $u : \Omega \to \Gamma$
- Generalizes:
  - Dynamical Optimal Transport: special case $\Omega \subseteq \mathbb{R}$ + interpolation
  - Method of calibrations: special case $\Gamma \subseteq \mathbb{R}$
- Vector-valued domain & range
- Natural extension to $\nabla^2 u, \Delta u$
- More dimensions = better sledding!

Medical Imaging with Deep Learning

Lübeck, 7 - 9 July 2021

2021.midl.io

mic.uni-luebeck.de

`lellmann@mic.uni-luebeck.de`