Example Sheet 1

The difficulty of these exercises varies considerably. Exercises marked with + are relatively easy but useful to recapitulate the various definitions, while those marked ++ require at least one clever idea. Exercises marked +++ are not for the faint-hearted. You should at least attempt the exercises that are marked with an exclamation mark (!).

Exercise 1 (Definition of Convexity) +

Complete the proofs of Thm. 3.5 (Jensen), Prop. 3.10, and Prop. 3.15.

Exercise 2 (Convex Sets) +

Show that the following sets are convex:

a) For a given set of $x^1, \ldots, x^m \in \mathbb{R}^n$, $m \geqslant 2$, and any $j \in \{1, \ldots, m\}$, the set of points so that their distance to x^j is not greater than to every of the remaining points (the *Voronoi cell* for x^j):

$$V = \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, m\} : \|x - x^j\|_2 \leqslant \|x - x^i\|_2\}.$$

b) The *Lorentz* cone

$$K = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} | \ \|x\| \leq t\}$$

for any given norm in $\|\cdot\|$ in $\mathbb{R}^n$ (usually the 2-norm is implicitly referred to).

c) The sum $Y = Y_1 + Y_2$, where

$$Y_1 = \{x \in \mathbb{R}^2 \ | \|x\|_2 < 1/4\}, \quad Y_2 = \{x \in \mathbb{R}^2 \ | |x_1 - 1| \leqslant 1, |x_2| \leqslant 1\}.$$

Exercise 3 (!) (Simplices) ++

We call the convex hull $\Delta_n = \operatorname{con}\{v^0, v^1, \ldots, v^m\}$ an $(m\text{-})$*simplex* in $\mathbb{R}^n$ iff the v^i are *affinely independent*, i.e., if the only choice of $\lambda_i \in \mathbb{R}$ such that $\lambda_0 v^0 + \ldots + \lambda_m v^m = 0$ and $\lambda_0 + \ldots + \lambda_m = 0$ is $\lambda_i = 0$ for all i. Show that every $x \in \Delta_n$ can be *uniquely* represented as a convex combination of the v^i.

Exercise 4 (!) (Improper Functions) ++

Characterize the set of lower semi-continuous, convex functions that are *not* proper.

Guide: Start from the set of points where the function has finite value.

Exercise 5 (!) (Basic Convexity, Convexity on Lines) +

a) Show that the following functions are convex together with their respective domains (i.e., they are convex on all of $\mathbb{R}^n$ when extended to $+\infty$ outside of their domain): $f_1 : \mathbb{R}_{>0} \to \mathbb{R}$, $f_1(x) = \frac{1}{x}$; $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = \exp(x)$; $f_3 : \mathbb{R}_{>0} \to \mathbb{R}$, $f_3(x) = -\log(x)$; $f_4 : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$, $f_4(x, y) = \frac{x^2}{y}$; $f_5 : \mathbb{R}^{n \times m} \to \mathbb{R}$, $f_5(X) = \|X\|_\sigma$, where $\|\cdot\|_\sigma$ is the spectral norm; $f_6 : \{X \in \mathbb{R}^{n \times n} | X \text{ symmetric}\} \to \mathbb{R}$, $f_6(X) = \lambda_{\max}(X)$, where $\lambda_{\max}(X)$ is the maximal eigenvalue of X.

b) Show that $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is convex if and only if the function $g : \mathbb{R} \to \bar{\mathbb{R}}$,

$$g(t) := \begin{cases} f(x + tv), & x + tv \in \operatorname{dom} f \\ +\infty, & x + tv \notin \operatorname{dom} f \end{cases}$$

is convex for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

c) Prove or disprove: If f is convex, then f is continuous on $\operatorname{dom} f$.

Exercise 6 (Geometric and Arithmetic Mean) +

Show the inequality of the geometric and arithmetic mean:

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i$$

for $x_1, \ldots, x_n \geqslant 0$.

Guide: Use exercise 5.

Exercise 7 (Derivative Tests) ++

Show the following theorem: Assume $C \subseteq \mathbb{R}^n$ is open and convex, and $f : C \to \mathbb{R}$ is differentiable. Then the following conditions are equivalent:

1. f is convex,
2. $\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geqslant 0 \quad \forall x, y \in C$,
3. $f(x) + \langle \nabla f(x), y - x\rangle \leqslant f(y) \quad \forall x, y \in C$,
4. if f is additionally twice differentiable: $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

Guide: Consider one-dimensional sections of f.

Exercise 8 (!) (Carathéodory) ++

Show that, for any set $X \subseteq \mathbb{R}^n$, every element $x \in \operatorname{con} X$ can be written as a convex combination of at most $n+1$ elements of X.

Exercise 9 (!) (Convex Hulls of Compact Sets) ++

Let $X \subset \mathbb{R}^n$ be a compact set. Show that $\operatorname{con} X$ is compact as well. Can we say something similar without assuming boundedness, i.e., are convex hulls of closed sets also closed?

Guide: Use Carathéodory's theorem.

Exercise 10 (!) (Semidefinite Cone) +

a) Show that the set K_n^{SDP} of symmetric positive semidefinite matrices in $\mathbb{R}^{n\times n}$ is a pointed closed convex cone.

b) Show that the function

$$f(X) = \begin{cases} -\log\det X^{-1}, & X \in K_n^{SDP}, \\ +\infty, & \text{otherwise} \end{cases}$$

is convex.