Mathematical Tripos Part II: Michaelmas Term 2014

Numerical Analysis – Lecture 8

Problem 2.23 (The advection equation) A useful paradigm for hyperbolic PDEs is the *advection equation*

$$u_t = u_x, \qquad 0 \le x \le 1, \qquad t \ge 0, \tag{2.10}$$

where $u = u(x,t)$. It is given with the initial condition $u(x,0) = \varphi(x)$, $x \in [0,1]$ and (for simplicity) the boundary condition $u(1,t) = \varphi(t+1)$. The exact solution of (2.10) is simply $u(x,t) = \varphi(x+t)$, a unilateral shift leftwards. This, however, does not mean that its numerical modelling is easy.

Example 2.24 (Instability) We commence by semidiscretizing $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}[u_{m+1}(t) - u_{m-1}(t)]$, so coming to the ODE $u'_m(t) = \frac{1}{2h}[u_{m+1}(t) - u_{m-1}(t)]$. For the Euler method, the outcome is

$$u_m^{n+1} = u_m^n + \tfrac{1}{2}\mu(u_{m+1}^n - u_{m-1}^n), \qquad m = 0...M, \quad n \in \mathbb{Z}_+,$$

with $u_0^n = 0$ for all n. In matrix form this reads

$$\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n, \qquad A = \begin{bmatrix} 1 & \frac{1}{2}\mu & & \\ -\frac{1}{2}\mu & 1 & \ddots & \\ & \ddots & \ddots & \frac{1}{2}\mu \\ & & -\frac{1}{2}\mu & 1 \end{bmatrix}.$$

The matrix A is normal, with the eigenvalues $\lambda_\ell = 1 + i\mu\cos\ell\pi h$ (see Example 2.16), so that $\|A\|^2 = 1 + \mu^2$, hence instability for any μ.

Method 2.25 (Upwind method) If we semidiscretize $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{h}[u_{m+1}(t) - u_m(t)]$, and solve the ODE again by Euler's method, then the result is

$$u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n), \qquad m = 0...M, \quad n \in \mathbb{Z}_+ \tag{2.11}$$

The local error is $\mathcal{O}(k^2 + kh)$ which is $\mathcal{O}(h^2)$ for a fixed μ, hence convergence if the method is stable.

The eigenvalue analysis of stability does not apply here, since the matrix A in $\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n$ is no longer normal (see Example 2.17), so we do it directly (as in Lecture 5). We let the boundary condition at $x = 1$ be zero and define $\|\boldsymbol{u}^n\| = \max_m |u_m^n|$. It follows at once from (2.11) that

$$\|\boldsymbol{u}^{n+1}\| = \max_m |u_m^{n+1}| \le \max_m \{|1-\mu|\,|u_m^n| + \mu\,|u_{m+1}^n|\} \le (|1-\mu| + \mu)\|\boldsymbol{u}^n\|, \qquad n \in \mathbb{Z}_+.$$

Therefore, $\mu \in (0,1]$ means that $\|\boldsymbol{u}^{n+1}\| \le \|\boldsymbol{u}^n\| \le \cdots \le \|\boldsymbol{u}^0\|$, hence stability.

Matlab demo: Download the Matlab GUI for *Solving the Advection Equation, Upwinding and Stability* from `http://www.maths.cam.ac.uk/undergrad/course/na/ii/advection/advection.php` and solve the advection equation (2.10) with the different methods provided in the demonstration. Experience what can go wrong when "winding" in the wrong direction!

Method 2.26 (The leapfrog method) We semidicretize (2.10) as $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}[u_{m+1}(t) - u_{m-1}(t)]$, but now solve the ODE with the second-order *midpoint rule*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_{n-1} + 2k\boldsymbol{f}(t_n, \boldsymbol{y}_n), \qquad n \in \mathbb{Z}_+ .$$

The outcome is the two-step *leapfrog* method

$$u_m^{n+1} = \mu\,(u_{m+1}^n - u_{m-1}^n) + u_m^{n-1}. \tag{2.12}$$

The error is now $\mathcal{O}(k^3 + kh^2) = \mathcal{O}(h^3)$. We analyse stability by the Fourier technique, assuming that we are solving a Cauchy problem. Thus, proceeding as before,

$$\widehat{u}^{n+1}(\theta) = \mu\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right)\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) \tag{2.13}$$

whence

$$\widehat{u}^{n+1}(\theta) - 2\mathrm{i}\mu\, \sin\theta\, \widehat{u}^n(\theta) - \widehat{u}^{n-1}(\theta) = 0, \qquad n \in \mathbb{Z}_+\,,$$

and our goal is to determine values of μ such that $|\widehat{u}^n(\theta)|$ is uniformly bounded for all n, θ. This is a difference equation $w_{n+1} + bw_n + cw_{n-1} = 0$ with the general solution $w_n = c_1\lambda_1^n + c_2\lambda_2^n$, where λ_1, λ_2 are the roots of the characteristic equation $\lambda^2 + b\lambda + c = 0$, and c_1, c_2 are constants, dependent on the initial values w_0 and w_1. If $\lambda_1 = \lambda_2$, then solution is $w_n = (c_1 + c_2 n)\lambda^n$. In our case, we obtain

$$\lambda_{1,2}(\theta) = \mathrm{i}\mu\sin\theta \pm \sqrt{1 - \mu^2\sin^2\theta}\,.$$

Stability is equivalent to $|\lambda_{1,2}(\theta)| \le 1$ for all θ and this is true if and only if $\mu \le 1$.

Problem 2.27 (Stability in the presence of boundaries) It is easy to extend Fourier analysis for the Euler method $u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n)$, with the initial condition $u(x,0) = \phi(x)$, $x \in [0,1)$, and zero boundary condition along $x = 1$. Consider the Cauchy problem for the advection equation with the initial condition $u(x,0) = \phi(x)$ for $x \in [0,1)$, and $u(x,0) = 0$ otherwise (it isn't differentiable, but this is not much of a problem). Solving the Cauchy problem with Euler, we recover $\boldsymbol{u}^n$ that is identical to the solution obtained from the zero boundary condition. This justifies using Fourier analysis for the problem with a boundary, and we obtain

$$\widehat{u}^{n+1}(\theta) = H(\theta)\,\widehat{u}^n(\theta)\,, \qquad H(\theta) = (1-\mu) + \mu e^{\mathrm{i}\theta}\,,$$

so that $\max|H(\theta)| = |1-\mu| + |\mu|$, hence stability if and only if $\mu \le 1$.

Unfortunately, this is no longer true for leapfrog. Closer examination reveals that we cannot use leapfrog at $m = 0$, since u_{-1}^n is unknown. The naive remedy, setting $u_{-1}^n = 0$, leads to instability, which propagates from the boundary inwards. We can recover stability letting, for example, $u_0^{n+1} = u_1^n$ (the proof is *very* difficult).

Matlab demo: The Matlab GUI for *Stability of 1D PDEs* features different choices of boundary conditions. A brief description of those is given at the bottom of the page `http://www.maths.cam.ac.uk/undergrad/course/na/ii/pde_stability/pde_stability.php`. Go and see how the solution to the diffusion- or wave equation changes when changing the boundary conditions. Do you face any stability problems in those cases?

Problem 2.28 (The wave equation) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad x \in [0,1], \qquad t \ge 0,$$

given with initial (for u and u_t) and boundary conditions. The usual approximation looks as follows

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu(u_{m+1}^n - 2u_m^n + u_{m-1}^n)\,,$$

with the Courant number being now $\mu = k^2/h^2$.

To advance in time we have to pick up the numbers $u_m^1 = u(x_m, k)$ (of course they should depend on the initial derivative $u_t(x,0)$. Euler's method provides the obvious choice $u(x_m,k) = u(x_m,0) + ku_t(x_m,0)$, but the following technique enjoys better accuracy. Specifically, we set u_m^1 to the right-hand side of the formula

$$\begin{aligned} u(x_m,k) &\approx u(x_m,0) + ku_t(x_m,0) + \tfrac{1}{2}k^2 u_{tt}(x_m,0) \\ &= u(x_m,0) + ku_t(x_m,0) + \tfrac{1}{2}k^2 u_{xx}(x_m,0) \\ &\approx u_m^0 + \tfrac{1}{2}\mu(u_{m-1}^0 - 2u_m^0 + u_{m+1}^0) + ku_t(x_m,0)\,. \end{aligned}$$

The Fourier analysis (for Cauchy problem) provides

$$\widehat{u}^{n+1}(\theta) - 2\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) = -4\mu\sin^2\tfrac{\theta}{2}\,\widehat{u}^n(\theta)\,,$$

with the characteristic equation $\lambda^2 - 2(1 - 2\mu\sin^2\frac{\theta}{2})\lambda + 1 = 0$. The product of the roots is one, therefore stability (that requires the moduli of both λ to be at most one) is equivalent to the roots being complex conjugate, so we require

$$(1 - 2\mu\sin^2\tfrac{\theta}{2})^2 \le 1.$$

This condition is achieved if and only if $\mu = k^2/h^2 \le 1$.