Dr J Lellmann

Mathematical Tripos Part II: Michaelmas Term 2014

Numerical Analysis – Lecture 13

Problem 3.13 (The Poisson equation) We consider the *Poisson equation*

$$\nabla^2 u = f, \quad -1 \le x, y \le 1, \tag{3.11}$$

where f is analytic and obeys the periodic boundary conditions

$$f(-1, y) = f(1, y), \quad -1 \le y \le 1, \qquad f(x, -1) = f(x, 1), \quad -1 \le x \le 1.$$

Moreover, we add to (3.11) the following *periodic boundary conditions*

$$\begin{aligned} u(-1, y) &= u(1, y), \quad u_x(-1, y) = u_x(1, y), \quad -1 \le y \le 1 \\ u(x, -1) &= u(x, 1), \quad u_y(x, -1) = u_y(x, 1), \quad -1 \le x \le 1. \end{aligned} \tag{3.12}$$

With these boundary conditions alone a solution of (3.11) is only defined up to an additive constant. Hence, we add a *normalisation condition* to fix the constant:

$$\int_{-1}^{1} \int_{-1}^{1} u(x, y)\, dx\, dy = 0. \tag{3.13}$$

We have the spectrally convergent Fourier expansion

$$f(x, y) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l} e^{i\pi(kx+ly)}$$

and seek the Fourier expansion of u

$$u(x, y) = \sum_{k,l=-\infty}^{\infty} \hat{u}_{k,l} e^{i\pi(kx+ly)}.$$

Since

$$0 = \int_{-1}^{1} \int_{-1}^{1} u(x, y)\, dx\, dy = \sum_{k,l=-\infty}^{\infty} \hat{u}_{k,l} \int_{-1}^{1} \int_{-1}^{1} e^{i\pi(kx+ly)}\, dx\, dy = \hat{u}_{0,0},$$

and

$$\nabla^2 u(x, y) = -\pi^2 \sum_{k,l=-\infty}^{\infty} (k^2 + l^2) \hat{u}_{k,l} e^{i\pi(kx+ly)},$$

together with (3.11), we have

$$\begin{cases} \hat{u}_{k,l} = -\dfrac{1}{(k^2 + l^2)\pi^2} \hat{f}_{k,l}, \quad k, l \in \mathbb{Z}, \ (k, l) \ne (0, 0) \\ \hat{u}_{0,0} = 0. \end{cases}$$

Matlab demo: See *FFT with Spectral Methods* at `http://www.maths.cam.ac.uk/undergrad/course/na/ii/poisson_equation/poisson_equation.php` for an implementation of the above numerical method for solving (3.11). In order to compute a solution to Poisson's equation download the m-files from the bottom of the page and run poisson_equation.m with Matlab. You can change the default forcing term f by changing g from $\sin(xpi)\sin(ypi)$ to any other periodic function you are interested in.

Remark 3.14 Applying a spectral method to the Poisson equation is not representative for its application to other PDEs. The reason is the special structure of the Poisson equation. In fact, $\phi_{k,l} = e^{i\pi(kx+ly)}$ are the eigenfunctions of the Laplace operator with

$$\nabla^2 \phi_{k,l} = -\pi^2 (k^2 + l^2) \phi_{k,l},$$

and they obey periodic boundary conditions.

Problem 3.15 (General second-order linear elliptic PDE) We consider the more general second-order linear elliptic PDE

$$\nabla^\top(a\nabla u) = f, \quad -1 \le x, y \le 1,$$

with $a > 0$, and a and f periodic. We again impose the periodic boundary conditions (3.12) and the normalisation condition (3.13). We can rewrite

$$\nabla^\top(a\nabla u) = \frac{\partial}{\partial x}(au_x) + \frac{\partial}{\partial y}(au_y) = a\nabla^2 u + a_x u_x + a_y u_y$$

and get

$$\begin{aligned}
-\pi^2 \left(\sum_{k,l=-\infty}^{\infty} \hat{a}_{k,l} e^{i\pi(kx+ly)}\right) \left(\sum_{k,l=-\infty}^{\infty} (k^2 + l^2)\hat{u}_{k,l} e^{i\pi(kx+ly)}\right) \\
-\pi^2 \left(\sum_{k,l=-\infty}^{\infty} k\hat{a}_{k,l} e^{i\pi(kx+ly)}\right) \left(\sum_{k,l=-\infty}^{\infty} k\hat{u}_{k,l} e^{i\pi(kx+ly)}\right) \\
-\pi^2 \left(\sum_{k,l=-\infty}^{\infty} l\hat{a}_{k,l} e^{i\pi(kx+ly)}\right) \left(\sum_{k,l=-\infty}^{\infty} l\hat{u}_{k,l} e^{i\pi(kx+ly)}\right) \\
= \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l} e^{i\pi(kx+ly)},
\end{aligned}$$

where

$$a(x,y) = \sum_{k,l=-\infty}^{\infty} \hat{a}_{k,l} e^{i\pi(kx+ly)}, \quad f(x,y) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l} e^{i\pi(kx+ly)}, \quad u(x,y) = \sum_{k,l=-\infty}^{\infty} \hat{u}_{k,l} e^{i\pi(kx+ly)}$$

In the next steps we replace the products by convolutions (using the bivariate version of (3.4)), truncate the expansions to $-N/2+1 \le k, l \le N/2$ and impose the normalisation condition $\hat{u}_{0,0} = 0$. This results in a system of $N^2 - 1$ linear algebraic equations in the unknowns $\hat{u}_{k,l}$, $k, l = -N/2+1, \ldots, N/2$, $(k,l) \ne (0,0)$:

$$\begin{aligned}
-\pi^2 \sum_{m,n=-N/2+1}^{N/2} \big[\hat{a}_{k-m,l-n}(m^2+n^2)\hat{u}_{m,n} + (k-m)\hat{a}_{k-m,l-n} m\hat{u}_{m,n} \\
+(l-n)\hat{a}_{k-m,l-n} n\hat{u}_{m,n}\big] = \hat{f}_{k,l}.
\end{aligned}$$

Matlab demo: See the online documentation for *Spectral Methods for the Poisson Equation* at `http://www.maths.cam.ac.uk/undergrad/course/na/ii/poisson_equation/poisson_equation.php` for an exemplar Matlab code of how this can be implemented.

Discussion 3.16 (Analyticity and periodicity) The fast convergence of spectral methods rests on two properties of the underlying problem: analyticity and periodicity. If one is not satisfied the rate of convergence in general drops to polynomial. However, to a certain extent, we can relax these two assumptions while still retaining the substantive advantages of Fourier expansions.

- *Relaxing analyticity*: In general, the speed of convergence of the truncated Fourier series of a function f depends on the smoothness of the function. In fact, the smoother the function the faster the truncated series converges, i.e., for $f \in C^p(-1,1)$ we receive an $\mathcal{O}(N^{-p})$ order of convergence.

  Spectral convergence can be recovered, once analyticity is replaced by the requirement that $f \in C^\infty(-1,1)$, i.e., $f^{(m)}(x)$ exists for all $x \in (-1,1)$ and $m = 0, 1, 2, \ldots$. Consider, for instance, $f(x) = e^{-1/(1-x^2)}$. Then, $f \in C^\infty(-1,1)$ but cannot be extended analytically because of essential singularities at ± 1. Nevertheless, one can show that $|\hat{f}_n| \sim \mathcal{O}(e^{-cn^\alpha})$, where $c > 0$ and $\alpha \approx 0.44$. While this is slower than exponential convergence in the analytic case (cf. Remark 3.7), it is still faster than $\mathcal{O}(n^{-m})$ for any integer m and hence, we have spectral convergence.
- *Relaxing periodicity*: Disappointingly, periodicity is necessary for spectral convergence. Once this condition is dropped, we are back to the setting of Theorem 3.3, i.e., Fourier series converge as $\mathcal{O}(N^{-1})$ unless $f(-1) = f(1)$. One way around this is to change our set of basis functions, e.g., to Chebyshev polynomials.