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Mathematical Tripos Part II: Michaelmas Term 2014

Numerical Analysis – Lecture 14

Revision 3.17 (Chebyshev polynomials) Let $T_n(x) = \cos(n \arccos x)$, $n \geq 0$. Each T_n is a polynomial of degree n, i.e.,

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ \ldots,$$

and is called the nth *Chebyshev polynomial* (of the first kind). They form a sequence of orthogonal polynomials, which are orthogonal with respect to the weight function $(1-x^2)^{-1/2}$ in $(-1,1)$. In fact, we have

$$\int_{-1}^{1} T_m(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & m = n = 0 \\ \frac{\pi}{2}, & m = n \geq 1,\ m,n \in \mathbb{Z} \\ 0, & m \neq n, \end{cases} \tag{3.14}$$

(where this can be proven by letting $x = \cos\theta$ and using the identity $T_n(\cos\theta) = \cos n\theta$). Moreover, the sequence T_n obeys the three-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n = 1, 2, \ldots.$$

Method 3.18 (Non-periodic problems and Chebyshev methods) Since $\{T_n\}_{n=0}^{\infty}$ form an orthogonal sequence we can expand a general integrable function f in

$$f(x) = \sum_{n=0}^{\infty} \breve{f}_n T_n(x), \tag{3.15}$$

with coefficients $\breve{f}_n$, $n = 0, 1, 2, \ldots$. Multiplying (3.15) by $T_m(x)\,(1-x^2)^{-1/2}$ and integrating for $x \in (-1,1)$ yields

$$\int_{-1}^{1} f(x)T_m(x)\,\frac{dx}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \breve{f}_n \int_{-1}^{1} T_n(x)T_m(x)\,\frac{dx}{\sqrt{1-x^2}}.$$

Further, using the orthogonality property (3.14) results in an explicit expression for the coefficients

$$\breve{f}_0 = \frac{1}{\pi}\int_{-1}^{1} f(x)\,\frac{dx}{\sqrt{1-x^2}}, \quad \breve{f}_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}},\ n = 1, 2, \ldots.$$

Next, letting $x = \cos\theta$ we obtain

$$\int_{-1}^{1} f(x)T_n(x)\quad\frac{dx}{\sqrt{1-x^2}} \quad = \quad \int_{0}^{\pi} f(\cos\theta)T_n(\cos\theta)\quad d\theta \quad = \quad \frac{1}{2}\int_{-\pi}^{\pi} f(\cos\theta)\cos n\theta \quad d\theta$$

The connection to Fourier expansions is apparent, given that $\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta})$. More precisely, the Fourier transform of a function g defined in the interval $[a,b]$, $a < b$, and which is periodic with period $b-a$, is given by the sequence

$$\hat{g}_n = \frac{1}{b-a}\int_a^b g(\tau)e^{-\frac{2\pi i n\tau}{b-a}}\,d\tau,\ n \in \mathbb{Z}.$$

In particular, letting $g(x) = f(\cos x)$ and $[a,b] = [-\pi,\pi]$, we have

$$\hat{g}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\tau)e^{-in\tau}\,d\tau,\ n \in \mathbb{Z}.$$

Therefore,

$$\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(\hat{g}_{-n} + \hat{g}_n)$$

and we deduce that

$$\breve{f}_n = \begin{cases} \hat{g}_0, & n = 0 \\ \hat{g}_{-n} + \hat{g}_n, & n = 1, 2, \ldots. \end{cases}$$

Discussion 3.19 For a general integrable function f the computation of its Chebyshev expansion is equivalent to the Fourier expansion of the function $g(x) = f(\cos x)$. Since the latter is periodic with period 2π, we can use a DFT to compute $\breve{f}_n$ and hence keep the benefits of periodic functions. In particular, if f can be analytically extended, then $\breve{f}_n$ decays spectrally fast for $n \gg 1$. Hence, the Chebyshev expansion inherits the rapid convergence of spectral methods without ever assuming that f is periodic.

Method 3.20 (The algebra of Chebyshev expansions) Let $\mathcal{B}$ be the set of analytic functions in $[-1, 1]$ that can be extended analytically into the complex plane. We identify each such function with its Chebyshev expansion. Like the set $\mathcal{A}$, the set $\mathcal{B}$ is a linear space and is closed under multiplication. In particular, we have

$$\begin{aligned} T_m(x)T_n(x) &= \cos(m \arccos x) \cos(n \arccos x) \\ &= \frac{1}{2}\left[\cos((m-n)\arccos x) + \cos((m+n)\arccos x)\right] = \frac{1}{2}[T_{|m-n|}(x) + T_{m+n}(x)] \end{aligned}$$

and hence,

$$\begin{aligned} f(x)g(x) = \sum_{m=0}^{\infty} \breve{f}_m T_m(x) \cdot \sum_{n=0}^{\infty} \breve{g}_n T_n(x) &= \frac{1}{2} \sum_{m,n=0}^{\infty} \breve{f}_m \breve{g}_n [T_{|m-n|}(x) + T_{m+n}(x)] \\ &= \frac{1}{2} \sum_{m,n=0}^{\infty} \breve{f}_m (\breve{g}_{|m-n|} + \breve{g}_{m+n}) T_n(x). \end{aligned}$$

Lemma 3.21 (Derivatives of Chebyshev polynomials) *The derivatives of Chebyshev polynomials can be expressed explicitly as the linear combinations*

$$T'_{2n}(x) = 4n \sum_{l=0}^{n-1} T_{2l+1}(x), \tag{3.16}$$

$$T'_{2n+1}(x) = (2n+1)T_0(x) + 2(2n+1) \sum_{l=1}^{n} T_{2l}(x). \tag{3.17}$$

Proof. We prove (3.16). The proof for (3.17) follows similar arguments. We have $T'_{2n}(x) = 2n \sin(2n \arccos x)(1 - x^2)^{-1/2}$, and hence $T'_{2n}(\cos\theta) \sin\theta = 2n \sin(2n\theta)$. On the other hand

$$4n \sin\theta \sum_{l=0}^{n-1} T_{2l+1}(\cos\theta) = 4n \sum_{l=0}^{n-1} \cos(2l+1)\theta \sin\theta = 2n \sum_{l=0}^{n-1} (\sin(2l+2)\theta - \sin 2l\theta) = 2n \sin 2n\theta.$$

□

Remark 3.22 (Application to PDEs) With Lemma 3.21 all derivatives of u can be expressed in an explicit form as a Chebyshev expansion (cf. Exercise 19 on Example Sheets). For the computation of the Chebyshev coefficients the function f has to be sampled at the so-called Chebyshev points $\cos(2\pi k/N)$, $k = -N/2 + 1, \ldots, N/2$. This results into a grid, which is denser towards the edges. For elliptic problems this is not problematic, however for initial value PDEs such grids can cause numerical instabilities.