Dr J Lellmann

Mathematical Tripos Part II: Michaelmas Term 2014

Numerical Analysis – Examples' Sheet 3

21. As discussed in the Lectures periodicity is necessary for spectral convergence. Suppose that an analytic function f on $[-1,1]$ is not periodic, yet $f(-1)=f(+1)$ and $f'(-1)=f'(1)$. Integrating by parts the Fourier coefficients $\hat{f}_n$ show that $\hat{f}_n=\mathcal{O}(n^{-3})$. Using the analysis of Lecture 11 show that the rate of convergence of the N-terms truncated Fourier expansion of f is hence $\mathcal{O}(N^{-2})$.

    Now, suppose $f(-1)\neq f(1)$. We can force the values at the endpoints to be equal. Set $f(x)=\frac{1}{2}(1-x)f(-1)+\frac{1}{2}(1+x)f(+1)+g(x)$, where $g(x)=f(x)-\frac{1}{2}(1-x)f(-1)-\frac{1}{2}(1+x)f(+1)$. Verify that $g(\pm1)=0$ and that if f is analytic then so is g. The idea is now to represent f as a linear function plus the Fourier expansion of g, i.e.,

    $$f(x)=\frac{1}{2}(1-x)f(-1)+\frac{1}{2}(1+x)f(+1)+\sum_{n=-\infty}^{\infty}\hat{g}_n\, e^{i\pi nx}.$$

    We can iterate this idea: To do so, construct a function h for which $h(\pm1)=h'(\pm1)=0$ and verify that $\hat{h}_n=\mathcal{O}(n^{-3})$. [*Hint: In the second construction the function f will be represented as a cubic function plus the Fourier expansion of h.*]

22. Consider the following boundary value problem for the heat equation

    $$\begin{cases} u_t=u_{xx}, & -1\le x\le 1,\ t>0\\ u(-1,t)=u(1,t),\ u_x(-1,t)=u_x(1,t), & t>0\\ u(x,0)=e^{i\pi Mx}, & -1\le x\le 1.\end{cases},$$

    where $M\in\mathbb{Z}$. By separation of variables one can compute the exact solution and get

    $$u(x,t)=e^{-M^2t^2}\ e^{i\pi Mx}.$$

    Now, approximate the solution u by its N-term truncated Fourier series and solve the spectral approximation for the heat equation, i.e.,

    $$\sum_{n=-N/2+1}^{N/2}\frac{d\hat{u}_n}{dt}(t)e^{i\pi nx}=\sum_{n=-N/2+1}^{N/2}\hat{u}_n(t)\frac{d^2}{dx^2}e^{i\pi nx}.$$

    What do you receive? What is the error of this method with the correct choice of N?

23. Prove that the Gauss-Seidel method for the solution of $A\boldsymbol{x}=\boldsymbol{b}$ converges whenever the matrix A is symmetric and positive definite. Show, however, by a 3×3 counterexample, that the Jacobi method for such an A need not converge. [*Warning: For Jacobi, it is not enough to construct a positive definite A such that $2D-A$ is not positive definite, because we did not prove that the Householder-John theorem gives a criterion. So, you need also to prove that $\rho(D^{-1}(A-D))>1$.*]

24. Let the Gauss-Seidel method be applied to the equations $A\boldsymbol{x}=\boldsymbol{b}$ when A is the nonsymmetric 2×2 matrix

    $$A=\begin{bmatrix}10 & -3\\ 3 & 1\end{bmatrix}.$$

    Find the spectral radius of the iteration matrix. Then show that the relaxation method, described in Lecture 17, can reduce the spectral radius by a factor of 2.9. Further, show that iterating twice with Gauss-Seidel with this relaxation decreases the error $\|\boldsymbol{x}^{(k)}-\boldsymbol{x}^{(\infty)}\|$ by more than a factor of ten. Estimate the number of iterations of the original Gauss-Seidel method that would be required to achieve this decrease in the error.

25. The function $u(x)=x\,(x-1)$, $0\le x\le1$, is defined by the equations $u''(x)=2$, $0\le x\le1$, and $u(0)=u(1)=0$. A difference approximation to the differential equation provides the estimates $u_m\approx u(mh)$, $m=1,2,\ldots,M-1$, through the system of equations

    $$\begin{cases}u_{m-1}-2u_m+u_{m+1}=2h^2, & m=1,2,\ldots,M-1\\ u_0=u_M=0,\end{cases},$$

$h = 1/M$, and M is a large positive integer. Show that the exact solution of the system is just $u_m = u(mh)$, $m = 1, 2, \ldots, M-1$.

We employ the notation $u_m^{(\infty)} = u(mh)$, because we let the system be solved by the Jacobi iteration, using the starting values $u_m^{(0)} = 0$, $m = 1, 2, \ldots, M-1$. Prove that the iteration matrix H has the spectral radius $\rho(H) = \cos(\pi/M)$. Further, by regarding the initial error vector $\boldsymbol{u}^{(0)} - \boldsymbol{u}^{(\infty)}$ as a linear combination of the eigenvectors of H, show that the largest component of $\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(\infty)}$ for large k is approximately $(8/\pi^3)\cos^k(\pi/M)$. Hence deduce that the Jacobi method requires about $2.5M^2$ iterations to achieve $\|\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(\infty)}\|_\infty \le 10^{-6}$.

26. The function $u(x,y) = 18x(1-x)y(1-y)$, $0 \le x, y, \le 1$, is the solution of the Poisson equation $u_{xx} + u_{yy} = 36(x^2 + y^2 - x - y) = f(x,y)$, say, subject to u being zero on the boundary of the unit square. We pick $h = 1/6$ and seek the solution of the five-point equations

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = h^2 f(mh, nh), \quad 1 \le m \le 5,\ 1 \le n, \le 5,$$

where $u_{m,n}$ is zero if (mh, nh) is on the boundary of the square. Let the multigrid method be applied, using only this fine grid and a coarse grid of mesh size $1/3$, and let every $u_{m,n}$ be zero initially. Calculate the 25 residuals of the starting vector on the fine grid. Then, following the *restriction* procedure in the lecture notes, find the residuals for the initial calculation on the coarse grid. Further, show that if the equations on the coarse grid are solved exactly, then the resultant estimates of u at the four interior points of the coarse grid all have the value $5/6$. By applying the *prolongation operator* to these estimates, find the 25 starting values of $u_{m,n}$ for the subsequent iterations of Gauss-Seidel or Jacobi on the fine grid. Further, show that if one Jacobi iteration is performed, then $u_{3,3} = 23/24$ occurs, which is the estimate of $u(1/2, 1/2) = 9/8$.

Matlab demo: Download the Matlab GUI for *Multigrid Methods* at `http://www.maths.cam.ac.uk/undergrad/course/na/ii/multigrid/multigrid.php`. There the Poisson equation is solved in one space dimension with a forcing term that represents a range of frequencies. Try running the GUI for the three featured methods, relaxed Jacobi, Gauss-Seidel and multigrid, and discuss the error decrease as a function of the number of iterations.

27. Apply the standard form of the conjugate gradient method to the linear system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

starting as usual with $\boldsymbol{x}^{(0)} = \mathbf{0}$. Verify that the residuals $\boldsymbol{r}^{(0)}, \boldsymbol{r}^{(1)}$ and $\boldsymbol{r}^{(2)}$ are mutually orthogonal, that the search directions $\mathbf{d}^{(0)}, \mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ are mutually conjugate, and that $\boldsymbol{x}^{(3)}$ satisfies the equations.

28. Let the standard form of the conjugate gradient method be applied when A is positive definite. Express $\mathbf{d}^{(k)}$ in terms of $\boldsymbol{r}^{(i)}$ and $\beta^{(i)} > 0$, $i = 0, 1, \ldots, k$. Then deduce in a few lines from the formula $\boldsymbol{x}^{(k+1)} = \sum_{i=0}^{k} \omega^{(i)} \mathbf{d}^{(i)}$, from $\omega^{(i)} > 0$, and from the fact that $\boldsymbol{r}^{(i)}$ are orthogonal, that the sequence $\{\|\boldsymbol{x}^{(k)}\| : k = 0, 1, \ldots\}$ increases monotonically.

29. The polynomial $p(x) = x^m + \sum_{i=0}^{m-1} c_i x^i$ is the *minimal polynomial* of the $n \times n$ matrix A if it is the polynomial of lowest degree that satisfies $p(A) = 0$. Note that $m \le n$ holds because of the Cayley-Hamilton theorem.

Give an example of a 3×3 symmetric positive definite matrix with a quadratic minimal polynomial.

Prove that (in exact arithmetic) the conjugate gradient method requires at most m iterations to calculate the exact solution of $A\boldsymbol{v} = \boldsymbol{b}$, where m is the degree of the minimal polynomial of A.