Dr J Lellmann

Mathematical Tripos Part II: Michaelmas Term 2014

Numerical Analysis – Examples' Sheet 4

30. Let A be an $n \times n$ TST matrix such that $a_{1,1} = \alpha$ and $a_{1,2} = \beta$. Show that the Jacobi iteration for solving $Ax = b$ converges if $2|\beta| < |\alpha|$. Moreover, prove that if convergence is required for all $n \geq 1$ then this inequality is necessary as well as sufficient.

31. Let A be an $n \times n$ TST matrix with $a_{k,k} = \alpha$ and $a_{k,k+1} = a_{k+1,k} = \beta$. Verify that $\alpha \geq 2|\beta| > 0$ implies that the matrix is positive definite. Now, we precondition the conjugate gradient method for $A\boldsymbol{x} = \boldsymbol{b}$ with the Toeplitz lower-triangular bidiagonal matrix Q,
$$q_{k,l} = \begin{cases} \gamma, & k = l \\ \delta, & k = l+1, \\ 0, & \text{otherwise.} \end{cases}$$
Determine real numbers γ and δ such that QQ^T differs from A in just the $(1,1)$ coordinate. Prove that with this choice of γ and δ the preconditioned conjugate gradient method converges in just two iterations.

32. Let
$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}, \quad S = \begin{pmatrix} A_1 & O \\ O & A_3 \end{pmatrix},$$
where A_1, A_3 are symmetric $n \times n$ matrices and the rank of the $n \times n$ matrix A_2 is $r \leq n-1$. We further stipulate that the $(2n) \times (2n)$ matrix A is positive definite. Let $A_1 = Q_1 Q_1^T$, $A_3 = Q_3 Q_3^T$ be Cholesky factorizations and assume that the preconditioner Q is the lower-triangular Cholesky factor of S (hence $QQ^T = S$). Prove that
$$B = Q^{-1} A Q^{-T} = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix}, \quad \text{where} \quad F = Q_1^{-1} A_2 Q_3^{-T}.$$
Supposing that the eigenvalues of B are $\lambda_1, \ldots, \lambda_{2n}$, while the eigenvalues of FF^T are $\mu_1, \ldots, \mu_n \geq 0$, prove that, without loss of generality,
$$\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{n+k} = 1 + \sqrt{\mu_k}, \quad k = 1, 2, \ldots, n.$$
Prove that the rank of FF^T is at most r, thereby deducing that B has at most $2r+1$ distinct eigenvalues. What does this tell you about the number of steps before the preconditioned conjugate gradient method terminates in exact arithmetic?
Matlab demo: Download the Matlab GUI for *Preconditioning of Conjugate Gradient* from `http://www.maths.cam.ac.uk/undergrad/course/na/ii/precond/precond.php`. Setup an example for a system matrix A of the type just discussed and use the GUI to compute the eigenvalues of $A^T A$ and of the preconditioned matrix. How does the number of iterations the CG method needs changes? How robust is the CG method to perturbations of A or b by a small random matrix or vector respectively?

33. Let A be the 3×3 matrix
$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$
where λ is real and nonzero. Find an explicit expression for A^k, $k = 1, 2, 3, \ldots$.
The sequence $\boldsymbol{x}^{(k+1)}$, $k = 0, 1, 2, \ldots$, is generated by the power method $\boldsymbol{x}^{(k+1)} = A\boldsymbol{x}^{(k)} / \|A\boldsymbol{x}^{(k)}\|$, where $\boldsymbol{x}^{(0)}$ is a nonzero vector in $\mathbb{R}^3$. Deduce from your expression for A^k that the second and third components of $\boldsymbol{x}^{(k+1)}$ tend to zero as $k \to \infty$. Further, show that this remark implies $A\boldsymbol{x}^{(k+1)} - \lambda \boldsymbol{x}^{(k+1)} \to \mathbf{0}$, so the power method tends to provide a solution to the eigenvalue equation.
Matlab demo: Reproduce your findings using the Matlab GUI for *Computing eigenvalues and eigenvectors* from `http://www.maths.cam.ac.uk/undergrad/course/na/ii/eigenstuff/eigenstuff.php`. How does the situation change when you change one of the λ-entries in A to another value?

34. Let A be a symmetric 2×2 matrix with distinct eigenvalues and normalized eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Given $\boldsymbol{x}^{(0)} \in \mathbb{R}^2$, the sequence $\boldsymbol{x}^{(k+1)}$, $k = 0, 1, 2, \ldots$, is generated in the following way. The *Rayleigh*

quotient $\lambda_k = \boldsymbol{x}^{(k)} A\boldsymbol{x}^{(k)}/\|x^{(k)}\|^2$ is taken as an estimate for an eigenvalue of A, the vector norm being Euclidean. Then, inverse iteration gives

$$\boldsymbol{y} = (A - \lambda_k I)^{-1}\boldsymbol{x}^{(k)}, \quad \text{and we set } \boldsymbol{x}^{(k+1)} = \boldsymbol{y}/\|\boldsymbol{y}\|.$$

Show that, if $\boldsymbol{x}^{(k)} = (\boldsymbol{v}_1 + \epsilon_k \boldsymbol{v}_2)/(1+\epsilon_k^2)^{1/2}$, where $|\epsilon_k|$ is small, then $|\epsilon_{k+1}|$ is of magnitude $|\epsilon_k|^3$. In other words, the method enjoys a *third order* rate of convergence.

35. The symmetric matrix

$$A = \begin{pmatrix} 9 & -8 & 2 \\ -8 & 9 & -2 \\ 2 & -2 & 10 \end{pmatrix} \quad \text{has the eigenvector} \quad \boldsymbol{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Calculate an orthogonal matrix Ω by a Householder transformation such that $\Omega\boldsymbol{v}$ is a multiple of the first coordinate vector $\boldsymbol{e}_1$. Then, form the product $\Omega^T A \Omega$. You should find that this matrix is suitable for deflation. Hence, identify all the eigenvalues and eigenvectors of A.

36. Show that the vectors $\boldsymbol{x}$, $A\boldsymbol{x}$ and $A^2\boldsymbol{x}$ are linearly dependent in the case

$$A = \begin{pmatrix} 4 & 5 & 2 & 0 \\ -26 & -14 & 1 & 4 \\ -2 & 2 & 3 & 1 \\ -43 & -8 & 13 & 9 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}.$$

Hence, calculate two of the eigenvalues of A. Obtain by deflation a 2×2 matrix whose eigenvalues are the remaining eigenvalues of A. Then, find the other eigenvalues of A.

37. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix

$$A = \begin{pmatrix} 9 & -1 & 2 & 2 \\ -1 & 3 & 4 & 2 \\ 2 & 4 & 14 & -3 \\ 2 & 2 & -3 & 4 \end{pmatrix}.$$

Your final matrix should be symmetric and should have the same trace as A.

38. Let A be an $n \times n$ symmetric tridiagonal matrix that is not deflatable (i.e., all the elements of A that are adjacent to the diagonal are nonzero). Prove that A has n distinct eigenvalues. Prove also that, if A has a zero eigenvalue and a single iteration of the QR algorithm is applied to A, then the resultant tridiagonal matrix is deflatable. [*Hint: In the first part show that for each eigenvalue λ there is a unique solution to $A\boldsymbol{w} = \lambda\boldsymbol{w}$. In the second part deduce that a diagonal element of R is zero.*]

39. Let A be a 2×2 symmetric matrix whose trace does not vanish, let $A_0 = A$, and let the sequence of matrices $\{A_k : k = 1, 2, \ldots\}$ be calculated by applying the QR algorithm to A_0 (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of A_k. Show that, except in the special case when A is already diagonal, the sequence $\{(A_k)_{1,1} : k = 0, 1, \ldots\}$ converges monotonically to the eigenvalue of A of larger modulus. [*Hint: The sign of this eigenvalue is the same as the sign of the trace of A. Also, for any symmetric matrix B, we have $B_{1,1} = \boldsymbol{e}_1^T B \boldsymbol{e}_1$ and $\lambda_{min}\|\boldsymbol{x}\|^2 \leq \boldsymbol{x}^T B \boldsymbol{x} \leq \lambda_{max}\|\boldsymbol{x}\|^2$.*]

40. Apply a single step of the QR method to the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix},$$

where $\epsilon > 0$. You should find that the $(2,3)$ element of the new matrix is $\mathcal{O}(\epsilon^3)$ and that the new matrix has exactly the same trace as A.
Matlab demo: Download the Matlab GUI for *Visual QR* from `http://www.maths.cam.ac.uk/undergrad/course/na/ii/qr_hex/qr_hex.php` and let the QR method run for the matrix A above. Try it with your own choice of a square matrices A and see what the QR method is doing to the entries of A. What happens if you choose a symmetric matrix A, what if A is upper Hessenberg?

41. (For those who like analysis). Let A be a real 4×4 upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices A_k, $k = 0, 1, 2, \ldots$, are calculated from A by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{3,2}$ converges to zero as $k \to \infty$.